Second-Degree Discrimination with Advertising in Random Networks*

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Abstract

This paper proposes a framework of second-degree discrimination with two different versions of a service that are served over random networks with positive externalities. Consumers choose between purchasing a premium version of the service or a free version that comes with advertising about some product (unrelated to the service). The ads influence free version adopters' opinions and, through the induced effects on the product's sales, they affect the optimal pricing of the premium version. We relate the optimal pricing strategy to the underlying degree distribution and hazard rate function of the random network. When the platform in fact optimally chooses to discriminate by offering the two versions of the service, *hazard rate dominance* always implies higher optimal profits. The model provides foundations for empirical analysis since key features of social networks can be related to their degree distributions and hazard rate functions.

Keywords: Social networks, second-degree discrimination, advertising, degree distributions, hazard rate

JEL Classification: D83, D85, L1, M3

1 Introduction

We develop a model for exploring second-degree discrimination with advertising that is widely applicable to the provision of services over complex social networks in the presence of consumption externalities. Over the last decade, the expansion of the Internet, communication devices, and assorted technologies has dramatically increased the provision

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of online services over social networks.[1] Usually, the consumption of the service is more or less beneficial to consumers depending on whether or not their neighbors (e.g., other consumers, friends, or co-workers) are using it as well, Thus the platform could exploit these network externalities by implementing some type of price discrimination. Although some scenarios allow for the implementation of individualized prices as a function of the consumers' positions in the social network,[2] for many real-world complex social networks, perfect price discrimination becomes too costly, or simply impractical.

Instead, in many complex networks, platforms often resort to a particular type of second-degree discrimination where they offer two versions of their services: a *premium version*, which consumers purchase at a price, and a *free version*, where consumers pay no price but, in exchange, they receive ads of some other products from a variety of industries (in principle, unrelated to platform). Then, the platform receives a compensation from the company that advertises its product, which is usually based on the amount of ads served or on the profits from the advertising activity. An additional feature of these business models is that the premium version usually allows the consumers to enjoy the network externalities to a larger extent than the free version. We use the term *externality premium* to refer to this difference in the network externalities across both versions of the service. The goal of this paper is to explore optimal pricing decisions under this particular type of second-degree discrimination that is becoming extensively used in many social networks.

We assume that exposure to ads is costly for free version adopters which establishes the relevant trade-off to choose one version of the service or the other. On the other hand, advertising influences consumers' opinions about the advertised product, which affects the revenue from the sales of such a product. As a consequence, the revenue to the platform from the free version of the service and, hence, the optimal pricing of the premium version are also affected. To determine its optimal pricing strategy, the platform must weigh out the impact on its overall profits of two conflicting sources of revenue: (1) the revenue from the sales of the premium version of the service, which depends crucially on the characteristics of the network, and (2) the revenue from the sales of the advertised product, which depends largely on features of the advertised product market, such as demand conditions and production costs, and on how advertising affects consumers' valuations.

When dealing with complex networks, platforms can rely on the empirical statistic regularities that exhibit social networks.[3] Based on random networks, we provide some

[1]Providing services over social networks is the business model of platforms such as Google (general communication and information), Facebook and Twitter (social interaction), Whatsapp, Skype, and Line (communication), AirBnB (accommodation search), Waze (traffic and route forecasting), The Weather Channel (weather forecasting), Yelp and Foursquare (review and rating), YouTube, Vimeo, and Spotify (entertaining and information), Strava and MapmyRun (exercise and health tracking), Box (second-hand trading), or Tinder (dating).

[2]The pioneering papers that explore first-degree price discrimination by a monopolist over social networks are Candogan et al. (2012) and Bloch & Quérou (2013).

[3]See, e.g., Jackson (2008) where empirical regularities of social networks such as *small worlds*, *clus-*

general results about the optimal pricing of the service as a function of fundamentals, such as production costs, the externality premium, and of some key indicators of the degree distribution of the random network. Our model is able to capture some empirical regularities of second-degree discrimination with advertising in social networks.

Given that our main goal is the relationship between the platform and the consumers, this paper abstracts from the plausible relations that may exist between the platform and another company whose product is being advertised through the online service provision. Using a reduced form, we model both companies as being perfectly integrated and acting as a single monopolist that offers both commodities, the service and the advertised product.[4] For tractability, the analysis assumes that the cost to the consumers from their exposure to ads is relatively small, which directly implies that each consumer purchases either the premium or the free version of the service.

Our notion of advertising builds upon the *informative advertising* approach pursued by Lewis & Sappington (1994) and Johnson & Myatt (2006). As in these models, we consider that advertising informs consumers about the product's characteristics and helps them improve their knowledge about their true underlying preferences for the product.

Random networks are usually best interpreted from a dynamic perspective where the *underlying degree distribution* determines how the average neighborhood size evolves over time. In our model, the role of the random network in the platform's optimal pricing can be pinpointed by the degree distribution that generates the network and by its associated *hazard rate function*. On the one hand, the shape of the degree distribution directly determines the demands of both versions of the service. Other things equal, because of the externality premium, consumers with relatively large neighborhoods (or that expect relatively large neighborhoods in the future, for dynamic interpretations of our model) prefer the premium version rather than the free one. As a consequence, the demand of the premium version is positively related with the probability that the network generates relatively large neighborhoods. On the other hand, the shape of the hazard rate function is useful to determine whether or not increasing the service price raises the profits from the service sales and, through this channel, it plays an important role in optimal pricing (Propositions 1 and 2).

Including advertising into the platform's discrimination strategy leads to a more subtle analysis than that involving mere second-degree price discrimination. For instance, when advertising can induce a relatively high valuation for the advertised product, it naturally becomes profitable for the platform to raise the price of the service in order to increase the number of its free version adopters since these are the only consumers receiving ads. If this effect is sufficiently high, then raising the service price can be profitable even in those random networks where higher prices lowers the revenue from the premium version

tering, or *assortativity* are discussed at length.

[4] Our main insights would follow qualitatively if we considered instead a model with two different companies where the platform receives a compensation proportional to the profits from the advertised product sales.

sales. Taking into account the effects on the overall profits, both from the advertised product sales and from the premium service sales, Proposition 2 provides a condition under which the platform wishes to raise the price of the service when advertising raises the consumers' valuations of the product. Other things equal, increasing hazard rates, relative to decreasing hazard rates, make it easer for the required condition to hold.

On the key issue of the role of the social network in the platform's optimal profits, we derive a sufficient condition in terms of a ranking which is stronger than the classical *first order stochastic dominance*. We show in Proposition 3 that *hazard rate dominance* implies higher optimal profits for the platform, provided that in fact it optimally serves both versions of the service.

Our results provide empirical predictions on how the optimal pricing of the service depends on the shape of the degree distribution and on its associated hazard rate function. For dynamic interpretations of random networks, wherein neighborhood sizes evolve over time according to the corresponding degree distribution, decreasing hazard rates indicate that it becomes more likely for consumers to form new links as the network grows. This is an a intuitive idea that has recently been furnished with solid micro-founded grounds. Following a dynamic approach, Shin (2016) provides a theoretical characterization where decreasing (or constant) hazard rates arise if an only if nodes are more likely (or equally likely) to engage in new links as their degrees increase. Then, given the externality premium, decreasing hazard rates make consumers more willing to purchase the premium version of the service. Intuitively, with decreasing hazard, consumers expect a raise in the number of their neighbors and thus wish to benefit from the higher network externalities that the premium version provides. As a consequence, the platform finds beneficial to rely on the sales of the premium version of the service rather than on its advertising activity. Importantly, the empirical evidence suggests that most complex social networks adjust either to the pattern of *scale-free networks*, which are determined by a *power law* degree distribution (with decreasing hazard rates),[5] or to the *exponential* degree distribution pattern (with constant hazard rates).[6]

[5]In their seminal study of the World Wide Network, Barabási & Albert (1999) concluded that the degree distribution of nodes on the Internet adjusts to a power law distribution. More recently, Clauset et al. (2009) found empirical evidence that both the nodes on the Internet (at the level of autonomous systems) and the number of links in websites adjust to a power law distribution. Also, Stephen & Toubia (2009) argue that the emerging social commerce network through the Internet follows a typical power law distribution. In perhaps the largest structural analysis conducted up to date, Ugander et al. (2011) conclude that the Facebook social network features decreasing hazard rates as well, though it does not fit the pattern of a power law distribution. Also, decreasing hazard rates are always generated by two ubiquitous models of the theoretical literature on random networks: the *preferencial attachment model* proposed by Barabási & Albert (1999) and the *network-based search model* suggested by Jackson & Rogers (2007).

[6]See, e.g., Rosas-Calals et al. (2007) or Ghoshal & Barabási (2011).

1.1 Related Literature

To the best of our knowledge, this is the first paper that explores second-degree discrimination in social networks by means of a two-version service, one of the versions offering advertising. Our work is influenced by several fields of research in economics. At a general level, our article is related to the prolific literature, initiated by Farrell & Saloner (1985) and Katz & Shapiro (1985), that explores the effects of network externalities on economic decisions and to the classical second-degree discrimination analysis of Mussa & Rosen (1978) and Maskin & Riley (1984), wherein a monopolist offers a menu of different qualities of a single product.

Our approach to advertising owes much to the notion of informative advertising developed by Lewis & Sappington (1994) and Johnson & Myatt (2006).[7] The idea that advertising may generate costs to the consumers from their exposure to ads goes back to the seminal paper of Becker & Murphy (1993). Also, there is a profound connection between the effects of advertising that we consider and the structure of the endogenous information decision problem explored by Amir & Lazzati (2016) in the context of (common value) Bayesian supermodular games proposed by Van-Zandt & Vives (2007).[8] First, as in these games, consumers' valuations for the advertised product and advertising qualities are complements in our model. Given this, because of the informative role of advertising that we assume, the inverse demand of the advertised product rotates clockwise which, as in Johnson & Myatt (2006), leads to the key insight that the platform's profits are convex in the quality of the information transmitted through advertising. Although their problem has a different motivation, Amir & Lazzati (2016)'s assumption of an information structure that is convex in the supermodular order is conceptually similar to our demand rotation implications when the informative quality of advertising increases. In particular, in consonance with their main implication that Bayesian supermodular games only have extreme (pure strategy) equilibria, our paper also makes use of the result that the platform prefers an extreme choice on advertising, either completely non-informative or fully revealing.

Our technical analysis makes use of the literature on random networks that initiated with Erdös & Renyi (1959). Our dynamic interpretation of random networks meets the assumptions of the canonical *configuration model* which was originally proposed by Bender & Canfield (1978) and used subsequently by a number of influential papers in the

[7] In particular, our model incorporates the type of clockwise demand rotations proposed by Johnson & Myatt (2006) and exploits their result that, in a variety of circumstances, the monopolist's profits are convex in the amount of information that is transmitted through advertising. As in their paper, we obtain that, through its advertising activity, the platform optimally wishes either to reveal no information whatsoever or to be completely informative.

[8] Within the organizational literature, Dessein et al. (2016) explore an attention choice problem with analogous supermodular structure and results about the convexity of the objetive function in the information choice.

social networks area.[9] The recent micro-foundations by Shin (2016) on dynamic random networks provide a very interesting benchmark to interpret our results on the role hazard rate functions.

Less closely related, an influential branch of the literature has dealt with how companies can use the diffusion of information through social networks to increase their profits. The literature on optimal advertising in the presence of adoption externalities originated with Butters (1977) and Grossman & Shapiro (1994). Also, consumer targeting and advertising strategies through *word of mouth communication* have been explored by a number of papers since the seminal contributions of Ellison & Fudenberg (1995) and Bala & Goyal (1998).[10] Our paper departs from those contributions in two respects. First, the novel form of second-degree discrimination with advertising that our paper proposes is not studied by that branch of the literature. Secondly, our model does not consider that the information about the advertised product flows depending on how consumers are connected in the network. We assume that advertising is posted publicly to all free version adopters and the role of the network is to give consumers the incentives for their service adoption choices.

Recently, Gramstad (2016) has considered a monopolist that allows consumers to choose from a menu of differentiated products in the presence of local externalities for consumers that purchase the same product. While Gramstad (2016) explores the role of the network structure in optimal pricing in a context where consumers may choose between different versions of some product, as we do, there are important differences between both papers. The main one is that the role of advertising on the platform's profits is absent in his analysis. In addition, we consider that the size of the network externality depends only on the version chosen by the consumer himself and not on the versions chosen by his neighbors. As a consequence, Gramstad (2016)'s analysis focuses on segmented markets along the network, whereas we consider a single market with different versions of the product.

The rest of the paper is organized as follows. Section 2 proposes our new taxonomy of second-degree discrimination with advertising and lays out the model. Section 3 describes the paper's main results and Section 4 illustrates our model with some examples. Section 5 discusses equilibrium multiplicity and Section 6 concludes. All the proofs are relegated to the Appendix.

[9] See, e.g., Bollobás (2001), Newman et al. (2001), Jackson & Yariv (2007), Galeotti & Goyal (2009), and Fainmesser & Galeotti (2016)).

[10] For instance, Galeotti & Goyal (2009) relate optimal marketing strategies to the characteristics of the random network. Under the assumption that consumers only inform their neighbors if they themselves purchase the product, Campbell (2013) proposes a dynamic model of optimal pricing and advertising in random networks where information diffusion is endogenously generated. More recently, Fainmesser & Galeotti (2016) have built on the insights of Candogan et al. (2012) and Bloch & Quérou (2013) to explore pricing strategies when consumers are heterogenous with respect to their influence abilities, and to obtain key implications on welfare.

2 The Model

A platform sells two unrelated commodities to a unit mass of consumers, indexed by $i \in [0,1]$, that are embedded in a (complex) social network. The platform produces, on the one hand, $z \geq 0$ units of a consumption product, at a marginal cost $c > 0$. On the other hand, it also offers any discrete quantities of a two-version online service, at no cost, that it serves over the social network. One version of the service is offered with advertising about the product (*free version*) while the other is offered without advertising (*premium version*). By offering two different versions of the service, the platform implements a second-degree discrimination policy where the consumers decide which version they adopt.

Each consumer has a unit demand for the product and a unit demand for the service, and the expressions "purchase" or "adopt" a version of the service will be exchangeably used. Let z_i be consumer i's probability of purchasing the product and let x_i be the probability that consumer i adopts the premium version of the service. While premium version adopters pay a price $q \geq 0$ for each unit of the service, free version adopters incur an (exogenous) exposure cost $\psi > 0$ from the advertising activity.[11] Each consumer is willing to pay up to ω for a unit of the product and up to θ for a unit of the service. The consumers' valuations for the product ω are independently drawn from a uniform distribution $U[0,1]$ and their valuations for the service θ are independently drawn from some interval $(\underline{\theta}, \overline{\theta})$ according to some (common) distribution. The two commodities are totally unrelated and, therefore, the valuations ω and θ are assumed to be independent from each other.

2.1 Advertising the product

We consider that advertising is informative and, in particular, that it helps consumers to improve their knowledge of their tastes for the product. Specifically, we assume that consumers are uncertain about their valuation ω of the product. Prior to their purchasing decisions, they do not observe ω but receive some exogenous *public signal* $y \in [0,1]$. The signal y can be interpreted as the public observation of some posted information about the product quality, such as the one obtained from existing commercials, marketing samples, or from some other selling activities. Then, the advertising strategy of the platform consists of choosing an *advertising level* $a \in [0,1]$ that influences the consumers' beliefs about their valuations of the product. The mechanism through which consumers update their initial beliefs is very simple. The public signal and the advertising level are related to the consumers' valuations of the product according to the rule: $y = \omega$, with probability a, and y is an independent draw from $U[0,1]$, with probability $1-a$. Then, each consumer i

[11] Such type of costs from exposure to advertising have been commonly considered by the literature on advertising. See, e.g., Stigler & Becker (1977) and Becker & Murphy (1993).

obtains the posterior expectation of his valuation for the product by applying Bayes rule:

$$E[\omega \mid a] = ay + (1-a)(1/2). \tag{1}$$

Higher values of a indicate more informative advertising. If $a = 1$, the consumers believe that the signal y is their actual valuation for the product. If $a = 0$, the consumers obtain no information whatsoever from the advertising activity and retain their priors. In short, a describes the informative quality of the platform's advertising activity.

This formulation builds on the settings proposed by Lewis & Sappington (1994) and Johnson & Myatt (2006) to analyze informative advertising. In particular, our simple specification of signaling ads induces the type of fairly general clockwise rotations of demand proposed by Johnson & Myatt (2006) for an advertising activity. To see this, let us use $P_a(z)$ to denote the inverse demand of the product if *all* consumers receive an advertising level $a \in [0, 1]$ and suppose that the platform sells an amount z of the product. Since the product's valuation is drawn from a $U[0, 1]$ distribution, the platform will sell the product to those consumers receiving a public signal greater than $y = 1 - z$. Therefore, the platform must set a price equal to

$$P_a(z) = a(1-z) + (1-a)(1/2),$$

which gives us the expression for the induced inverse demand of the product conditioned on an advertising level a. It can be easily verified that the inverse demand $P_a(z)$ obtained above rotates clockwise (around the rotation point $(z^R, p^R) = (1/2, 1/2)$) as $a \in [0, 1]$ increases, exactly as stated in Definition 1 of Johnson & Myatt (2006). This yields useful implications for the optimal choice on advertising followed by the platform in our model. As we will show in Subsection 3.1, the platform's profits are convex in the advertising level $a \in [0, 1]$ so that it finds optimal to choose either $a^* = 0$ or $a^* = 1$. This is simply a direct consequence of the result obtained by Johnson & Myatt (2006) (Proposition 1), which establishes that the profits from the advertised product sales are quasi-convex in $a \in [0, 1]$ under the assumption that advertising rotates clockwise the induced inverse demand of the product.

At a more fundamental level, since the price $P_a(z)$ that the platform sets for the product (for a given quantity $z \in [0, 1]$) and the advertising level $a \in [0, 1]$ are complements, our advertising structure is also closely related to supermodularity approaches and, in particular, to the information acquisition problem explored by Amir & Lazzati (2016) in the context of (common value) Bayesian supermodular games. Under their supermodularity assumptions on the underlying information structure, they derive the key result of convexity of the value of information. As in Amir & Lazzati (2016), one can view our assumptions on advertising, and their immediate implications, as a set of conditions that exploits a supermodularity structure for a problem with an endogenous informative choice.

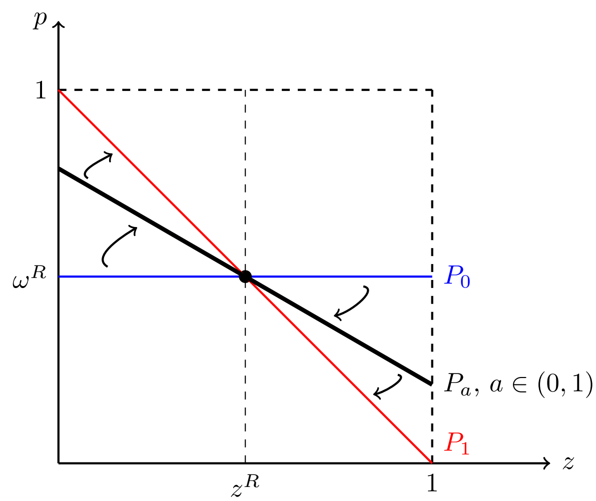

Figure 1: Rotation of Inverse Demand

2.2 The Strategies of the Platform and the Consumers

The platform and the consumers are engaged in a game where the platform chooses a *price* $p \geq 0$ *for the consumption product*, a *price* $q \geq 0$ *for (the premium version of) the service*, and an *advertising level* $a \in (0, 1]$, whereas the consumers make simultaneous *consumption decisions* about the service and the product $(z_i, x_i)_{i\in[0,1]}$. Price discrimination is not allowed for the advertised product so that all consumers face the common price p for the product, regardless of the version of the service that they adopt.

The advertising exposure cost is assumed to be very small relative to the valuation of the service, $\psi \in (0, \underline{\theta})$, which enables us to focus on equilibria where all consumers find optimal to purchase at least one version of the service (that is, at least the free version).[12] Given this, the analysis then explores the conditions under which some consumers might decide to purchase the premium version rather than the free one.

2.3 The Social Network and Preferences

We consider that the network's architecture is not affected by the consumers' decisions.[13] The social network allows consumers to interact locally with respect to their consumptions (only) of the service. In particular, the consumption of the service exhibits a local (positive) network effect: a consumer's utility from (any version of) the service increases

[12]Intuitively, we restrict attention to situations where all consumers are (at least) willing to adopt the free version despite of their exposure to ads.

[13]The underlying social network could in principle be a graph more general than the one generated by the links facilitated by the platform. For instance, the network could include also links provided by other platforms from different industries, and other links such as those from working relations, family, or friendship.

as his neighbors increase their consumptions.[14]

To capture the complexity of the social network, and to allow for dynamic interpretations of the model, we assume that it is stochastically generated. The platform and the consumers are uncertain about the specific configuration of the network but they commonly know the stochastic process that generates it. There is a set $[\underline{n}, \overline{n}] \subseteq \mathbb{R}_+$ of possible neighborhood sizes, or *degrees in the social network*.[15] Let $n_i \in [\underline{n}, \overline{n}]$ denote a possible *degree for consumer* i. The *degree distribution of the social network* is given by a twice continuously differentiable distribution $H(n)$, with a strictly positive density $h(n)$ over the support $[\underline{n}, \overline{n}]$. Thus, the degree of a fraction $H(n) = \int_{\underline{n}}^{n} h(m)dm$ of consumers does not exceed n.

Let $x(q)$ be the fraction of premium version adopters at price q, conditioned on the degree distribution $H(n)$. By considering a setting where each consumer adopts one of the versions of the service (which is ensured by the assumption $\psi < \underline{\theta}$), we trivially obtain that the average consumption of some version of the service of a consumer's neighbor equals one.[16] Therefore, the number n_i specifies the average consumption of the service of agent i's neighbors, conditioned on consumer i having degree n_i. In addition to this, the preference specification in (2) makes use of the *degree independence assumption* in order to guarantee that the only relevant information about the network for each consumer is his degree.[17] Given those considerations, the expected utility of a consumer i, conditioned on having degree n_i, is specified as

$$u(z_i, x_i \mid n_i) = \underbrace{z_i(\omega - p)}_{\text{product}} \;+\; \underbrace{x_i\big[\theta - q + (1+\beta)n_i\big]}_{\text{premium version of the service}} \;+\; \underbrace{(1-x_i)\big[\theta - \psi + n_i\big]}_{\text{free version of the service}}. \qquad (2)$$

The expression in (2) captures the presence of local (positive) network externalities. A consumer's utility raises by an amount of 1, when he adopts the free version of the service,

[14]In practice, these externalities take the form of informational gains (e.g., weather forecast, traffic monitoring, news services, or review and rating services), collaborative gains (e.g., online gaming or collaborative projects), and benefits from being able to interact with a higher number of people (e.g. second-hand trading, exercise tracking, or dating services).

[15]In some applications, one might consider $\underline{n} = 0$. Also, the set $[\underline{n}, \overline{n}]$ could be unbounded as well and, in particular, $\overline{n}$ could tend to infinity in some applications.

[16]Formally, under the assumptions of the *configuration model*, the expected consumption of some version of the service of a consumer i's neighbor can be computed as

$$\frac{\int_{\underline{n}}^{\overline{n}} m h_s(m)\big[\mathbb{P}(z_i = 1 \mid n_i = m) + \mathbb{P}(z_i = 0 \mid n_i = m)\big]dm}{\int_{\underline{n}}^{\overline{n}} n h_s(n)dn} = 1$$

since, given the assumption $\psi < \underline{\theta}$, we directly obtain $\mathbb{P}(z_i = 1 \mid n_i = m) + \mathbb{P}(z_i = 0 \mid n_i = m) = 1$ for each degree $m \in [\overline{n}, \underline{n}]$.

[17]The *degree independence assumption* states that the nodes of the network regard their shared links as independently chosen from the random network. This is a very common assumption in the literature on random networks and has been used, among others, by Jackson & Yariv (2007), Galeotti et al. (2010), Fainmesser & Galeotti (2016), and Shin (2016).

or by an amount $1 + \beta$, when he adopts the premium version, for each unit of the service consumed by his neighbors, regardless of the version that the neighbors adopt. We assume that $\beta > 0$, which formally gives us a single crossing condition for the two types of service adopters. The term β describes the presence of an *externality premium*, which seems to be the consistent with most real-world online services where the premium version allows consumers to enjoy the network externalities to a greater extent, compared to the free version.

From the preference specification in (2), it follows that the fraction of consumers that purchase the premium version of the service at price q, conditional on the degree distribution $H(n)$, is given by

$$\begin{aligned} x(q) &= \mathbb{P}(x_i = 1) = \mathbb{P}\big(\theta - q + (1+\beta)n_i \geq \theta - \psi + n_i\big) \\ &= \mathbb{P}\left(n_i \geq \frac{q-\psi}{\beta}\right) = 1 - H\left(\frac{q-\psi}{\beta}\right). \end{aligned} \tag{3}$$

For price q, let $n(q) = \frac{q-\psi}{\beta}$ be the cutoff degree such that $x_i = 0$ if $n_i \leq n(q)$ whereas $x_i = 1$ if $n_i > n(q)$. From the expression in (3) above, we observe that the fraction of consumers that purchase the premium version naturally decreases in its price q. The sensitivity of the demand $x(q)$ of the premium version with respect to its price depends on the random process that generates the network and on the externality premium. In particular, we have $x'(q) = -\left(\frac{1}{\beta}\right) h\big(n(q)\big)$. Finally, note that all consumers purchase the free version of the service (i.e., $x(q) = 0$) when its price satisfies $q \geq \overline{q}$, where the upper bound $\overline{q}$ on the service price is given by $\overline{q} = \beta\overline{n} + \psi$. Thus, in addition to the natural requirement that $q \in (\underline{\theta}, \overline{\theta})$, we need to consider that $q \leq \overline{q}$ throughout the analysis.

Hazard rate analysis, which has traditionally been used to compute "survival rates" in epidemiology and finance, turns out very useful to capture key features of how the random network evolves in dynamic interpretations of the model. Besides, in the current benchmark, the main insights on the platform's optimal pricing will be related to the hazard rate of the degree distribution. The *hazard rate function* of the random network with distribution degree $H(n)$ is the function on $[\underline{n}, \overline{n}]$ defined as

$$r(n) = \frac{h(n)}{1 - H(n)}. \tag{4}$$

For dynamic interpretations where the network evolves along several periods, the function $r(n)$ gives us the probability that a randomly selected consumer has approximately n neighbors in a subsequent period,[18] conditioned on his current neighborhood size being no less than n. Using a dynamic interpretation of the random generating process assumed in our model, Shin (2016) (Proposition 2) shows that increasing hazard rates follow if and

[18]Formally, for the continuous distribution case, $r(n)$ is the probability that the number of neighbors of a randomly selected consumer lies in the interval $(n - \varepsilon, n + \varepsilon)$, for $\varepsilon > 0$ sufficiently small.

only if a node is less likely to form additional new links as his degree increases. On the other hand, with decreasing hazard rates, the degree of a randomly chosen node in the network becomes arbitrarily large as the average neighborhood size increases (Shin (2016), Proposition 3).

3 Main Results

3.1 Optimal Advertising

Because of the assumed informative role of advertising, the profits from the sales of the advertised product are quasi-convex in $a \in [0, 1]$ so that the platform finds optimal to choose either $a^* = 0$ or $a^* = 1$. Specifically, since we are considering a unit mass of consumers and the consumers' valuations ω of the product follow a distribution $U[0, 1]$, a fraction

$$z_a(p) = E[\omega \,|\, a] - p$$

of consumers purchase the product at price p, conditional on receiving a (public) advertising signal y and an advertising level $a \in [0, 1]$. Recall that price discrimination is not allowed for the advertised product so that all consumers face the same price p for the product, regardless of the advertising level a to which they are exposed. Using the expression obtained in (1) for the consumers' posterior expectation, we obtain $z_a(p) = ay + (1 - a)(1/2) - p$. Then, for $a \in (0, 1]$, the platform's profits from the sales of the product to the free version adopters of the service are given by

$$\pi_a(p) = (p - c)\big[ay + (1 - a)(1/2) - p\big]. \tag{5}$$

These profits are linear in a and, if the public signal y satisfies the condition $y > 1/2$, then the platform will optimally choose $a^* = 1$. Otherwise, the platform would choose $a^* = 0$. As mentioned earlier, the function $\pi_a(p)$ in (5) above falls into the class of profit functions analyzed by Lewis & Sappington (1994) and Johnson & Myatt (2006) for informative advertising.[19] In order to explore second-degree discrimination, we will restrict attention to situations where the public signal y satisfies the condition $y \in (1/2, 1]$, so that the platform in fact discriminates across consumers by choosing an advertising level $a^* = 1$. Otherwise, no consumer would receive any information whatsoever from advertising and, therefore, the model would not allow for the study of second-degree discrimination. Accordingly, we will also consider $c \in (0, 1/2)$ to restrict attention to situations where the platform always obtains positive profits the sales of the product.

[19] In particular, Lewis & Sappington (1994) (Propositions 1 and 2) and Johnson & Myatt (2006) (Proposition 4) show that, even for more general production costs, the type of profit functions that we are analyzing in this paper turn out to be the maximum of convex functions so that they are themselves convex with respect to $a \in [0, 1]$. Therefore, the monopolist optimally chooses either $a^* = 0$ or $a^* = 1$. This has become a well-known result in the advertising literature with the interpretation that the monopolist wishes to target either a "niche" or a "mass" market.

3.2 Optimal Pricing

As derived in the previous subsection, the profits to the platform from the sales of the product, without ads (premium version of the service) and with ads (free version of the service) are given, respectively, by $\pi_0(p) = (p-c)(1/2-p)$ and $\pi_1(p) = (p-c)(y-p)$, where $c \in (0, 1/2)$ and $y \in (1/2, 1]$. Given a degree distribution $H(n)$ for the random network and a price $q \in (\underline{\theta}, \overline{\theta})$ (with $q \leq \overline{q}$) of the service, the platform wishes, on the one hand, to set a price $p^* \in (0,1)$ for the product so as to maximize the profits from the product sales

$$\pi\big(p, q\big) = x(q)\pi_0(p) + \big[1 - x(q)\big]\pi_1(p). \tag{6}$$

Since advertising affects (some of) the consumers' willingness to pay for the product, the profits from the product sales depend on the fraction $x(q)$ of premium version adopters. Furthermore, given an optimal price $p^* \in (0,1)$ of the product, the platform wishes to choose a price $q^* \in (\underline{\theta}, \overline{\theta})$, with $q^* \leq \overline{q}$, for the service so as to maximize its profits

$$\Pi(p^*, q) = qx(q) + \pi\big(p^*, q\big). \tag{7}$$

Formally, the optimal choices $\big((p^*, q^*, a^*), (z_i^*, x_i^*)_{i\in[0,1]}\big)$, with $a^* = 1$, correspond to a *Nash equilibrium* of the described game where the platform chooses the prices of the product and the service and consumers choose their purchases of the product (which determines the demand $z(p^*)$ of the advertised product) and self-select themselves to adopt one version or the other of the service (which determines the demand $x(q^*)$ of the premium version of the service). Existence of equilibrium is guaranteed since the profits specified in (6) and (7) are continuous functions on compact convex sets.

Proposition 1 describes the platform's optimal pricing strategy for interior prices.

Proposition 1. *Consider a random social network with degree distribution $H(n)$ and hazard rate function $r(n)$. Then, (i) the optimal price of the product is given by:*

$$p^* = \frac{(1/2 + c) + (y - 1/2)H\big(n(q^*)\big)}{2}; \tag{8}$$

(iia) if the interior service price $q^* \in \big(\underline{\theta}, \min\big\{\overline{\theta}, \overline{q}\big\}\big)$, with $x(q^*) \in (0,1)$, is an optimal choice by the platform, then it must satisfy the following first-order condition:

$$q^* = \frac{(y - 1/2)\big[(1/2 - c) + (y - 1/2)H\big(n(q^*)\big)\big]}{2} + \frac{\beta}{r\big(n(q^*)\big)}; \tag{9}$$

(iib) if, in addition to condition (1), the interior service price $q^* \in \big(\underline{\theta}, \min\big\{\overline{\theta}, \overline{q}\big\}\big)$ satisfies the second-order condition

$$\Big[(y - 1/2)^2 h\big(n(q^*)\big) - 4\beta\Big] h\big(n(q^*)\big) \leq \frac{2\beta h'\big(n(q^*)\big)}{r\big(n(q^*)\big)}, \tag{10}$$

then it corresponds to an optimal choice by the platform.

Interestingly, the concavity condition required in (10) for interior optimal prices might not be satisfied for each degree distribution that we consider. In particular, for degree distributions that feature decreasing densities ($h'(n) < 0$), the platform profits might be convex on the price q of the service. In such cases, the platform would optimally choose to provide only either the premium version or the free version of the service. For reasonable parameter values of the power law degree distribution, Example 3 illustrates that the advertising activity is relatively less profitable for the platform than the sales of the premium version of the service and, therefore, optimal pricing involves providing only the premium version of the service.

A raise in the price q that the monopolist sets for the service generates the following effects on its total profits:

1. Higher price q changes the profits from the service sales, $qx(q)$. The direction of this change depends on the shape of the degree distribution of the social network. In particular, an infinitesimal increase in q raises locally the profits from the service sales if and only if $r\big(n(q)\big) < 1/q$.

2. Higher price q always raises the optimal price p^* that the platform sets for the advertised product. In particular, it can be directly verified by the expression obtained in (8) that

$$\frac{\partial p^*(q)}{\partial q} = \frac{(y-1/2)h\big(n(q^*)\big)}{2\beta} > 0 \quad \text{at } q = q^*.$$

3. Higher price q lowers the fraction $x(q)$ of premium version adopters and, therefore, it raises the fraction

$$z(p^*) = x(q)z_0(p^*) + \big[1 - x(q)\big]z_1(p^*)$$

of consumers who purchase the product since $z_0(p^*) = (1/2 - p^*)$ and $z_1(p^*) = (y - p^*)$, which in our model directly implies that $z_1(p^*) > z_0(p^*)$ for each given p^*.

The result of the combination of the effects enumerated above on the platform's revenue is not obvious in general. The following straightforward corollary to Proposition 1 characterizes (locally) the sign of the derivative $\partial\Pi(p^*, q)/\partial q$.

Corollary 1. *Consider a random social network with degree distribution $H(n)$ and hazard rate function $r(n)$. Then, for each price q of the service such that $q \in \big(\underline{\theta}, \min\{\overline{\theta}, \overline{q}\}\big)$ and $x(q) \in (0,1)$, it follows that $\partial\Pi(p^*, q)/\partial q > 0$ if and only if the condition*

$$q < \frac{(y-1/2)\big[(y-c) - (y-1/2)x(q)\big]}{2} + \frac{\beta}{r\big(n(q)\big)}$$

is satisfied.

The existence of a solution q^* where the platform wishes to serve only the free version of the service follows when the condition stated in Corollary 1 is satisfied for each price $q \in \left(\underline{\theta}, \min\left\{\overline{\theta}, \overline{q}\right\}\right)$. This solution can either take the form of a corner solution, if we consider a finite maximum neighborhood size, or of an indeterminate explosive price, if we allow $\overline{n} \to +\infty$. This corresponds to situations where the platform is interested in making profits mainly through its advertising activity rather than relying on the sales of the premium version service.[20]

We turn now to explore the implications on the optimal service price q^* of an increase in the exogenous advertising signal y. The intuitive idea here is that lower costs of providing other sales or marking activities related to the product advertised by the platform, such as product samples, could raise the value of the signal y. In such cases, the consumers' perceptions about the value of the service improve for each fixed advertising level $a \in (0, 1]$. Proposition 2 provides the condition on the degree distribution and the hazard rate underlying the social network under which a rise in the advertising signal y increases the optimal price of the service.

Proposition 2. *Consider a random social network with degree distribution $H(n)$ and hazard rate function $r(n)$. Given an equilibrium where $q^* \in \left(\underline{\theta}, \min\left\{\overline{\theta}, \overline{q}\right\}\right)$ and $x(q^*) \in (0, 1)$, it follows that $\partial q^*/\partial y \geq 0$ if and only if the condition*

$$1 + \frac{r'\big(n(q^*)\big)}{\big[r\big(n(q^*)\big)\big]^2} \geq \frac{(y - 1/2)^2 h\big(n(q^*)\big)}{2\beta} \tag{11}$$

is satisfied.

We observe that, other things equal, social networks that feature increasing hazard rates facilite that the condition in (11) holds. On the other hand, the condition in (11) indicates that decreasing hazard rates with sufficiently high slopes (in absolute value), so that $r'\big(n(q^*)\big) < 0$ and $\left|r'\big(n(q^*)\big)\right|$ is sufficiently high, might lead the platform to optimally decrease the price of the service given a rise in y. A rise in the signal y makes advertising more profitable for the platform. In this case, the platform might raise its profits by relying (relatively) less on the profits from the service sales and, thus, by decreasing the price of the service in order to induce a higher proportion of free version adopters. However, following the key characterization result by Shin (2016) for a dynamic framework, wherein decreasing hazard rates appear if and only if a randomly picked consumer is more likely to enjoy a higher number of neighbors as the network grows, the platform has also incentives to raise the price of the service, given the externality premium. If this effect is sufficiently high, the platform might find beneficial to rely relatively more on the sales of the service even when advertising is able to induce higher valuations for the advertised product. Interestingly, many empirical studies suggest that decreasing hazard rates are present

[20] Recently, many firms (such as Twitter, YouTube, Facebook, or Google, just to mention a few of the most prominent ones) had adhered to this second scenario.

in most real-world complex, but relatively sparse, social networks (see, e.g., Barabási & Albert (1999), Clauset et al. (2009), Stephen & Toubia (2009) or Ugander et al. (2011)).

3.3 The Role of the Network on Optimal Profits

We now explore the central issue of how the random network structure influences the optimal overall profits $\Pi(p^*, q^*)$ that the platform obtains under its second-degree discrimination policy with advertising. Both the degree distribution and the hazard rate function are very useful to establish relations between different random social networks. To explore the role of the degree distribution that generates the network in optimal profits, we find convenient to consider a family of degree distributions $\{H_s(n)\}$ ($\{h_s(n)\}$) that is parameterized by some index $s \in [\underline{s}, \overline{s}]$. Accordingly, let us use $\Pi^*(s) = \Pi(p^*, q^*)$ to denote the platform optimal profits, conditional on a degree distribution $H_s(n)$. Then, the following definition states the classical notions of *first-order stochastic dominance* and *hazard rate dominance* in terms of local changes in the distributions.

Definition 1. *The family of degree distributions* $\{H_s(n)\}$ *is (i)* ordered by first-order stochastically dominance (FOSD–ordered) *if* $\partial H_s(n)/\partial s < 0$ *for each given* $n \in [\underline{n}, \overline{n}]$, *and is (ii)* ordered by hazard rate dominance (*HRD*–ordered) *if* $\partial r_s(n)/\partial s < 0$ *for each given* $n \in [\underline{n}, \overline{n}]$.

The *HRD* order gives us a stronger criterion than the *FOSD* order. Lemma 1 states formally this well-known relation in statistics about the two stochastic orders.

Lemma 1. *For a family of degree distributions* $\{H_s(n)\}$ *over the support* $[\underline{n}, \overline{n}]$, *it follows that*

$$\partial r_s(n)/\partial s < 0 \;\Rightarrow\; \partial H_s(n)/\partial s < 0$$

for each given $n \in [\underline{n}, \overline{n}]$.

Proposition 3 establishes that *HRD* for the degree distribution of the social network always increases the optimal profits of the platform when attention is restricted to interior optimal prices.

Proposition 3. *Suppose that the family of degree distributions* $\{H_s(n)\}$ *gives rise to interior optimal prices* $p^* \in (0, 1)$ *and* $q^* \in \left(\underline{\theta}, \min\left\{\overline{\theta}, \overline{q}\right\}\right)$, *with* $x(q^*) \in (0, 1)$. *Then, the platform's optimal profits* $\Pi^*(s)$ *are strictly increasing in* $s \in [\underline{s}, \overline{s}]$ *if the family* $\{H_s(n)\}$ *is* HRD*–ordered.*

The sufficient condition in Proposition 3 gives us a clear-cut criterion on how the network structure affects the platform's optimal profits in terms of an indicator of the degree distribution that can be estimated empirically. This might be useful for practitioners in the area. Consider two degree distributions $H_s(n)$ and $H_{s'}(n)$ such that $H_s(n)$

HRD $H_{s'}(n)$. It follows that $r_s(n) \leq r_{s'}(n)$ for each $n \in [\underline{n}, \overline{n}]$. The intuitive message for a dynamic interpretation of random networks, where the average neighborhood size increases over time, is that neighborhood sizes n that were already achieved in past periods are less likely to survive in future periods. In other words, as the network grows, the probability that the number of neighbors of a randomly chosen consumer raises in future periods is higher under $H_s(n)$ than under $H_{s'}(n)$. Then, given the externality premium $\beta > 0$, consumers value in average the premium version of the service relatively more under $H_s(n)$ than under $H_{s'}(n)$. This raises the platform's revenue from the service sales without affecting any of the fundamentals of the revenue from advertising (the production cost c or the signal y). In short, the hazard rate distribution $r_s(n)$ gives the platform more flexibility in its discrimination problem, which allows for higher optimal profits, relative to the optimal profits achievable under the hazard rate $r_{s'}(n)$.

4 Examples

This section presents a few examples using degree distributions which are relevant in the social networks literature. Our goal with these examples is to illustrate the platform's optimal discrimination strategy, as well as to obtain intuitions about how our main results work. For all the following examples, we consider $y = 3/4$, $c = 1/4$, $\psi = 0.01$, and $\beta = 1$.

Example 1 (Constant Hazard Rate: The Exponential Distribution). *For a degree support* $n \in [0, +\infty)$*, an* exponential degree distribution with parameter $\sigma > 0$ *is given by*

$$H(n) = 1 - e^{-n/\sigma} \quad \textit{and} \quad h(n) = (1/\sigma)e^{-n/\sigma}.$$

The corresponding hazard rate function is $r(n) = 1/\sigma$*. Exponential degree distributions are often used to capture the formation of links according to uniform randomness in growing random networks.*[21] *The result provided by Proposition 1 (iia) leads to that interior optimal prices* q^* *of the service must satisfy the condition*

$$2q^* = (2/16) - (1/16)e^{\frac{0.01 - q^*}{\sigma}} + 2\sigma.$$

As for the second-order condition stated in Proposition 1 (iib), notice that

$$\Big[(y - 1/2)^2 h(n) - 4\beta\Big] h(n) = \frac{1}{16\sigma^2} e^{-2n/\sigma} - \frac{4}{\sigma} e^{-n/\sigma}$$

and

$$\frac{2\beta h'(n)}{r(n)} = \frac{2}{\sigma^2} e^{-n/\sigma}$$

[21] See, e.g., Jackson (2008), Chapter 5, for an insightful description of the use of the exponential degree distribution in growing random networks.

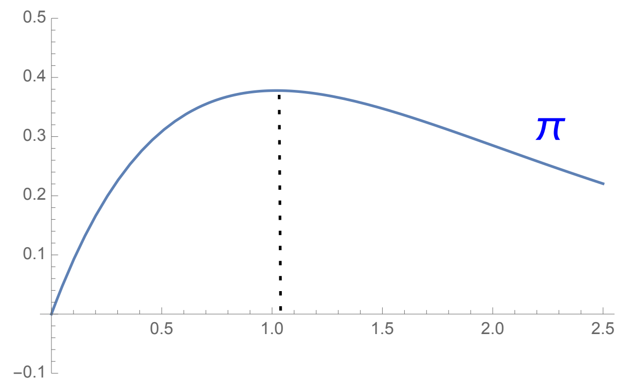

Figure 2: Platform's profits for the exponential degree distribution.

so that the sufficient requirement stated in (10) is automatically satisfied. Let us now focus on the exponential the exponential distribution with parameter $\sigma = 1$. *Using the equilibrium condition derived above, we obtain a unique equilibrium price* $q^* \approx 1.02$ *with optimal profits* $\Pi(p^*, q^*) \approx 0.37$. *The platform's profit function* $\Pi(p^*, q)$ *takes the form*

$$\Pi(p^*, q) = qe^{0.01-q} + \left[\frac{1}{8} - \frac{1}{8}e^{0.01-q}\right]^2,$$

which is depicted in Figure 2.

Example 2 (**Increasing Hazard Rate: The Erlang Distribution**). *For a degree support* $n \in [0, +\infty)$, *an* Erlang degree distribution[22] with parameters $\mu > 0$ and $\sigma \in \{1, 2, \dots\}$ *is given by the density*

$$h(n) = \frac{n^{\sigma-1}e^{-n/\mu}}{\mu^{\sigma}(\sigma-1)!}.$$

For this example, we will restrict attention to the parameter values $\mu = 1$ *and* $\sigma = 2$, *so that we obtain*

$$H(n) = 1 - (1+n)e^{-n} \quad \textit{and} \quad h(n) = ne^{-n}.$$

The corresponding hazard rate function is $r(n) = n/(1+n)$ *which increases in* n. *The result provided by Proposition 1 (iia) leads to that interior optimal prices* q^* *of the service must satisfy the condition*

$$2q^* = \frac{2}{16} - \frac{q^* + 0.99}{16}e^{0.01-q^*} + \frac{2(q^* + 0.99)}{q^* - 0.01}.$$

As for the second-order condition stated in Proposition 1 (iib), notice that

$$\left[(y - 1/2)^2 h(n) - 4\beta\right] h(n) = \left(\frac{n}{16}e^{-n} - 4\right) ne^{-n}$$

[22]Erlang distributions are special cases of the Gamma family of distributions.

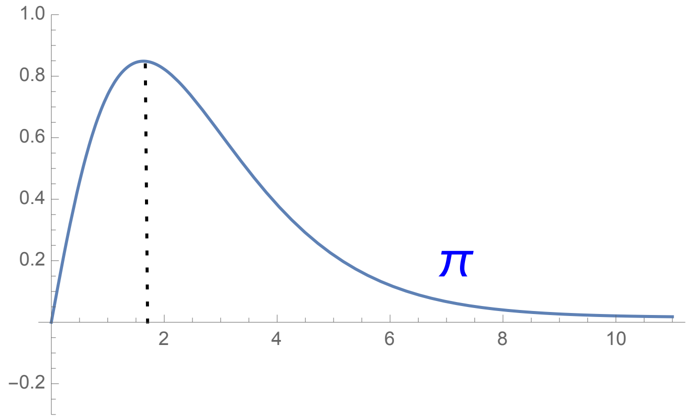

Figure 3: Platform's profits for the Erlang degree distribution.

and

$$\frac{2\beta h'(n)}{r(n)} = \frac{2(1-n^2)e^{-n}}{n}.$$

Then,

$$\Big[(y-1/2)^2 h(n) - 4\beta\Big] h(n) - \frac{2\beta h'(n)}{r(n)} = \left(\frac{n^2+32n-32}{16n}e^{-n} - 4\right) ne^{-n} \leq 0$$

for relatively low degrees n. Using the equilibrium condition derived above, we obtain a unique equilibrium price $q^* \approx 1.63$ with optimal profits $\Pi(p^*, q^*) \approx 0.85$. The platform's profit function $\Pi(p^*, q)$ takes the form

$$\Pi(p^*, q) = q(q+0.99)e^{0.01-q} + \left[\frac{1}{8} - \frac{1}{8}(q+0.09)e^{0.01-q}\right]^2,$$

which is depicted in Figure 3.

Example 3 (**Decreasing Hazard Rate: The Power Law Distribution**). *Empirical evidence suggests that most real-world online and Internet-based social networks are* scale-free *and that the corresponding degree distribution adjusts reasonably well to a* power law. *In addition, the literature on social networks identifies power law degree distributions as particularly suitable to model the formation of networks that follow a preferential attachment pattern. For a degree support $n \in [\underline{n}, +\infty)$, with $\underline{n} > 0$ a* power law degree distribution with parameter $\sigma > 1$*, is given by*

$$H(n) = 1 - \underline{n}^{\sigma-1} n^{-(\sigma-1)} \quad \textit{and} \quad h(n) = (\sigma-1)\underline{n}^{\sigma-1} n^{-\sigma}.$$

The corresponding hazard rate function is $r(n) = (\sigma-1)/n$, which decreases in n. The

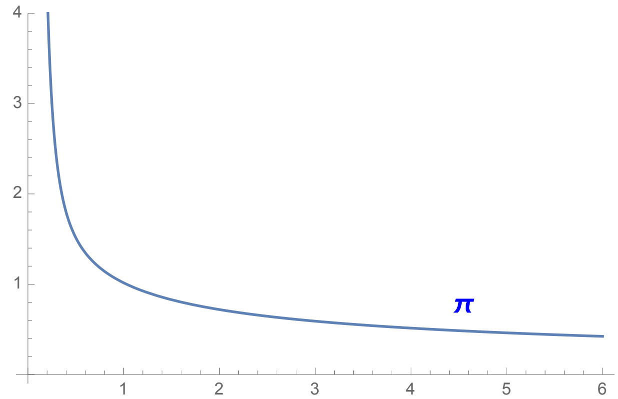

Figure 4: Platform's profits for the power law degree distribution.

second-order condition stated in Proposition 1 (iib) requires that

$$\Big[(y-1/2)^2 h(n) - 4\beta\Big] h(n) - \frac{2\beta h'(n)}{r(n)}$$
$$= \left(\frac{(\sigma-1)\underline{n}^{\sigma-1}n^{-\sigma}}{16} - 4\right)(\sigma-1)\underline{n}^{\sigma-1}n^{-\sigma} + \sigma\underline{n}^{\sigma-1}n^{-\sigma} \leq 0.$$

However, because the density of the degree distribution is always decreasing in its support, the inequality inequality is not satisfied for most values of σ. In particular the platform's profits turn out to be decreasing convex for most parameters of the power law degree distribution. Most empirical estimates propose values for the parameter σ that lie in the interval $(2,3)$. For this example, let us consider $\sigma = 5/2$ and $\underline{n} = 1$. Then, the platform's profit function $\Pi(p^*, q)$ takes the form

$$\Pi(p^*, q) = q(q-0.01)^{-3/2} + \left[\frac{1}{8} - \frac{1}{8}(q-0.01)^{-3/2}\right]^2,$$

which is depicted in Figure 4. The optimal choice is a corner solution where $q^* \to 0.01$. In this case, the platform prefers to rely on the revenue from the sales of the premium version of the service rather than on the revenue for the advertising activity. As a consequence, in this corner solution, the platform provides only the premium version of the service.

5 Equilibrium Multiplicity

Under our assumptions, there is always a (well-defined) solution for the platform's decision problem, provided that the second order condition in equation (10) is satisfied and the support $[\underline{n}, \overline{n}]$ of the degree distribution is bounded. Unfortunately, we cannot always guarantee uniqueness of equilibrium in our model (even for interior optimal prices). The main source of equilibrium multiplicity comes from the fact that the hazard rate of the degree distribution be either non-monotone or decreasing. For (weakly) increasing hazard rates, though, interior equilibrium, if it exists, is unique. Consider the function $\Psi(q)$ defined as

$$\Psi(q) = q - \frac{(y-1/2)\big[(1/2-c)+(y-1/2)H\big(n(q)\big)\big]}{2} - \frac{\beta}{r\big(n(q)\big)} \tag{12}$$

so that $\Psi(q^*) = 0$ is the equilibrium condition given in (1) for the service price. Since $H(n(q))$ and $h(n(q))$ are continuous functions in q, $\Psi(q)$ is continuous in $q \in \big(\underline{\theta}, \min\{\overline{\theta}, \overline{q}\}\big)$ as well. From the expression in (12), we directly observe that either non-monotone or decreasing hazard rates might lead to non-monotonicity of the function $\Psi(q)$ in the interval $\big(\underline{\theta}, \min\{\overline{\theta}, \overline{q}\}\big)$ and, therefore, to multiple optimal prices. Nevertheless, even when $\Psi(q)$ is non-monotone, a perturbation argument over the two parameters (ψ, β) ensures that the set of (interior) optimal prices q^* is *generically* a finite one. In particular, this perturbation argument allows us to eliminate equilibria where the consumers choose mixed strategies (i.e., the non-generic cases where $n_i = (q^* - \psi)/\beta$ for some $i \in [0,1]$) and where the equality $\Psi(q^*) = 0$ is satisfied by a continuum set of solutions q^*.

Using the function $\Psi(q)$ specified in (12), Proposition 4 shows that, under a mild requirement on the size of the externality premium, uniqueness of interior equilibrium is guaranteed if the hazard rate function is (weakly) monotone increasing.

Proposition 4. *Consider a random social network with degree distribution $H(n)$ and (weakly) monotone increasing hazard rate function $r(n)$. Assume that the externality premium β is no less than the lower bound $\underline{\beta}(y) \equiv (y-1/2)^2/2$. Suppose that the platform optimally chooses some interior price $q^* \in \big(\underline{\theta}, \min\{\overline{\theta}, \overline{q}\}\big)$, then such an optimal price q^* is unique.*

Using a model, different to the one proposed here, where a monopolist also chooses a price for a product that is distributed through a network with positive degree externalities, Shin (2016) obtains uniqueness of the optimal price when the underlying hazard rate is increasing. While increasing hazard rates directly ensure the required single-crossing property in his model, we also need an additional form of single-crossing property which, in our model, takes the form of the assumed strictly positive premium externality, $\beta > 0$.

6 Concluding Remarks

The growth of online and assorted technologies has made it possible for platforms that operate in social networks to implement advertising and discrimination strategies that were not available earlier. These new possibilities have motivated an array of important questions about how advertising and prices interact with the architecture, and the dynamic evolution, of social networks. In this article, we have suggested and investigated a new taxonomy of second-degree discrimination in social networks which is implemented through two versions of a product, one of them being offered with advertising.

We have obtained clear-cut predictions of how fundamentals, such as the difference in sizes of the network externalities or the advertising levels, affect the optimal discrimination strategy. Beyond that, this model says when platforms benefit from increasing their audiences for ads, depending on the shapes of the underlying hazard rate functions. The comparative statics of optimal profits with respect to *hazard rate dominance* is neat: hazard rate dominance always increases the optimal profits of the platform, provided that it optimally offers the two versions of the service.

There are different directions in which our model could be extended. First, we have considered the case of monopoly and it would be very interesting to explore the consequences of having several platforms competing in a common social network. Nonetheless, we believe our results continue to be compelling if, for most practical situations, we consider that there is indeed sufficient service differentiation and/or that different platforms do not actually serve a significant number of common links (i.e., there is a certain degree of network differentiation across platforms). Secondly, our framework assumes that the only factor that determines the size of the network externalities is the degree of the nodes. Thus, we have not incorporated important features present in real-world social networks (such as clustering or assortativity) which, in practice, could influence the consumption externalities. For example, the externality could be larger when common neighbors are shared or when the links are formed between consumers with some similar characteristics. Finally, our analysis has ignored the fact that the links could be used to transmit information about the characteristics of the advertised product. This information transmission would interact (possibly in a complementary or substitutive way) with the information distributed through formal advertising. This allows for interesting interactions between the formal advertising offered by the platform and the *world of mouth* "advertising" that spreads through the social network. In particular, for review and rating services, including this possibility into the analysis seems very important.

Finally, the current paper has restricted attention to a uniform distribution of the consumers' valuations of the product and has modeled advertising as a signaling device. It would be quite interesting to extend the type of second-degree discrimination proposed here to a benchmark with more general valuation distributions, as well as with a broader notion of advertising. Nonetheless, our framework is widely applicable in the dimension of the degree distribution, and it seems to capture empirical observations about how real-world business models operate in their provision of online services through social networks.

Appendix

Proof of Proposition 1. Consider a random network with degree distribution $H(n)$ and hazard rate function $r(n)$. Using the expressions in (6) and (7) for the platform's profits, we obtain

$$\begin{aligned}\Pi(p,q) &= qx(q) + x(q)\pi_0(p) + \big[1 - x(q)\big]\pi_1(p) \\ &= qx(q) + (p-c)\Big[x(q)(1/2-p) + \big[1-x(q)\big](y-p)\Big].\end{aligned}$$

Then, for a given price $q \in (\underline{\theta}, \min\{\overline{\theta}, \overline{q}\})$ of the service, we obtain the first-order condition with respect to the price p of the advertised product

$$\frac{\partial \Pi(p^*,q)}{\partial p} = \Big[x(q)(1/2-p) + \big[1-x(q)\big](y-p^*)\Big] - (p^*-c) = 0,$$

which implies

$$p^* = \frac{(y+c) - (y-1/2)x(q)}{2}. \tag{13}$$

Using the expression for $x(q)$ obtained in (3), the optimal price above can be rewritten as

$$p^* = \frac{(1/2+c) + (y-1/2)H\big(n(q)\big)}{2},$$

as stated in (i) of the proposition. The second-order condition for the price choice p is automatically satisfied since $\Pi(p,q)$ is strictly convex in p. Also, notice that $p^* \in (0,1)$ under our assumptions $y \in (1/2, 1]$ and $c \in (0, 1/2)$. Then, by plugging the expression derived in (13) into the platform's profit function, it follows that

$$\Pi(p^*,q) = qx(q) + \left[\frac{(y-c) - (y-1/2)x(q)}{2}\right]^2. \tag{14}$$

Given the optimal choice $p^* \in (0,1)$, the first-order condition with respect to the price q of the service leads to

$$\begin{aligned}\frac{\partial \Pi(p^*,q^*)}{\partial q} =& x(q^*) + \left[q^* - \frac{(y-1/2)\big[(y-c) - (y-1/2)x(q^*)\big]}{2}\right]x'(q^*) = 0 \\ \Leftrightarrow q^* =& \frac{(y-1/2)\big[(y-c)-(y-1/2)x(q^*)\big]}{2} - \frac{x(q^*)}{x'(q^*)}.\end{aligned} \tag{15}$$

Using the expression for $x(q)$ obtained in (3) and the definition of hazard rate function in (4), the optimal price derived above can be rewritten as

$$q^* = \frac{(y-1/2)\big[(1/2-c) + (y-1/2)H\big(n(q^*)\big)\big]}{2} + \frac{\beta}{r\big(n(q^*)\big)},$$

as stated in (iia) of the proposition. As for the second-order condition stated in (iib) of the proposition, first note that

$$\frac{\partial^2\Pi(p^*,q^*)}{\partial q^2} = \left[2 + \frac{(y-1/2)^2 x'(q^*)}{2}\right] x'(q^*) \\ + \left[q^* - \frac{(y-1/2)}{2}\big[(y-c)-(y-1/2)x(q^*)\big]\right] x''(q^*).$$

Furthermore, by using the first-order condition derived in (15), the expression above can be rewritten as

$$\frac{\partial^2\Pi(p^*,q^*)}{\partial q^2} = \left[2 + \frac{(y-1/2)^2 x'(q^*)}{2}\right] x'(q^*) - \frac{x(q^*)}{x'(q^*)} x''(q^*).$$

Therefore, by using the expression for $x(q)$ obtained in (3), the definition of hazard rate function in (4), and the direct implication that $x''(q^*) = -(1/\beta^2)h'\big(n(q^*)\big)$, we obtain that $\partial^2\Pi(p^*,q^*)/\partial q^2 \leq 0$ if and only if the stated condition

$$\Big[(y-1/2)^2 h\big(n(q^*)\big) - 4\beta\Big] h\big(n(q^*)\big) \leq \frac{2\beta h'\big(n(q^*)\big)}{r\big(n(q^*)\big)}$$

is satisfied. ∎

Proof of Proposition 2. Consider a random network with degree distribution $H(n)$ and hazard rate function $r(n)$. Define the function

$$\Gamma\big(q^*(y),y\big) = q^* - \frac{(y-1/2)\big[(1/2-c)+(y-1/2)H\big(n(q^*)\big)\big]}{2} - \frac{\beta}{r\big(n(q^*)\big)}$$

so that $\Gamma\big(q^*(y),y\big) = 0$ gives us the necessary condition for (interior) optimal prices of the service, as stated in Proposition 1 (iia). Then, application of the *implicit function theorem* allows us to obtain

$$\frac{\partial q^*}{\partial y} = -\frac{\partial\Gamma/\partial y}{\partial\Gamma/\partial q^*}$$

for $\partial\Gamma/\partial q^* \neq 0$. Thus, it can be verified that

$$\frac{\partial\Gamma}{\partial y} = -\frac{\big[(1/2-c)+(2y-1)H\big(n(q^*)\big)\big]}{2} < 0$$

for each $c \in (0,1/2)$ and each $y \in (1/2,1]$. On the other hand, it can be verified that

$$\frac{\partial\Gamma}{\partial q^*} = 1 - \frac{(y-1/2)^2 h\big(n(q^*)\big)}{2\beta} + \frac{r'\big(n(q^*)\big)}{\big[r\big(n(q^*)\big)\big]^2}.$$

Therefore, since $\partial\Gamma/\partial y < 0$, it follows that $\partial q^*/\partial y \geq 0$ if and only if $\partial\Gamma/\partial q^* \geq 0$ or, equivalently, if and only if the condition

$$1 + \frac{r'\big(n(q^*)\big)}{\big[r\big(n(q^*)\big)\big]^2} \geq \frac{(y-1/2)^2 h\big(n(q^*)\big)}{2\beta}$$

holds. ∎

Proof of Proposition 3. Consider a family of random networks with degree distributions $H_s(n)$ and hazard rate functions $r_s(n)$, parameterized using the index $s \in [\underline{s}, \overline{s}]$. Take a parameter value $s \in [\underline{s}, \overline{s}]$. By combining the necessary condition for (interior) optimal prices of the service obtained in Proposition 1 (iia) with the expression of the platform's profits obtained in (14), we can write the platform's optimal profits (for interior prices) as

$$\Pi(s) \equiv \Pi(p^*, q^*) = q^*\Big[1 - H_s\big(n(q^*)\big)\Big] + \left[\frac{\frac{\beta}{r_s(n(q^*))} + q^*}{y - 1/2}\right]^2.$$

Therefore,

$$\frac{\partial\Pi(s)}{\partial s} = -q^* \frac{\partial H_s\big(n(q^*)\big)}{\partial s} + \frac{2\left(\frac{\beta}{r_s(n(q^*))} + q^*\right)}{(y-1/2)^2}\left(-\frac{\beta\frac{\partial r_s(n(q^*))}{\partial s}}{\Big[r_s\big(n(q^*)\big)\Big]^2}\right). \tag{16}$$

Suppose that the family of distributions $\{H_s(n)\}$ is HRD–ordered. Note first that the term

$$\frac{2\left(\frac{\beta}{r_s(n(q^*))} + q^*\right)}{(y-1/2)^2}$$

is always strictly positive. Secondly, it follows from the definition of HRD dominance that $\partial r_s(n(q^*))/\partial s < 0$. Furthermore, using the the result stated in Lemma 1, we know that $\partial H_s(n(q^*))/\partial s < 0$ or, in other words, the family of distributions $\{H_s(n)\}$ is also FOSD–ordered. Then, using the expression in (16), it follows that $\partial\Pi(s)/\partial s > 0$. ∎

Proof of Proposition 4. Consider a random social network with degree distribution $H(n)$ and (weakly) monotone increasing hazard rate function $r(n)$. Using the expression in (12) for the function $\Psi(q)$, it directly follows that

$$\Psi'(q) = 1 - \frac{(y-1/2)^2 h\big(n(q)\big)}{2\beta} + \frac{r'\big(n(q)\big)}{\big[r\big(n(q)\big)\big]^2}. \tag{17}$$

Let $\underline{\beta}(y) \equiv (y-1/2)^2/2$ be the lower bound on the premium externality that guarantees that the term

$$1 - \frac{(y-1/2)^2 h\big(n(q)\big)}{2\beta}$$

is not negative for each $\beta \geq \underline{\beta}(y)$, for each value of the density $h\big(n(q)\big) \in (0,1)$. Therefore, if $\beta \geq \underline{\beta}(y)$ and $r'(n) \geq 0$, it follows from the expression in (17) that $\Psi'(q) \geq 0$. This condition ensures the (weakly) monotonicity of the function $\Psi(q)$, so that if the platform optimally chooses some interior price $q^* \in \big(\underline{\theta}, \min\left\{\overline{\theta}, \overline{q}\right\}\big)$, then it must be unique. ∎

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