Versioning with Advertising in Social Networks under Uniform Distributions of Valuations

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Abstract

This paper explores a second-degree discrimination scheme where a platform sells a two-version online service to a group of consumers engaged in a social network. The social network is randomly generated and features positive consumption externalities. Consumers choose between purchasing a premium version of the service, which enhances the network externalities, or a free version, which includes advertising about some other product—unrelated to the service—. The ads influence free version adopters' opinions and, through the induced effects on the advertised product's sales, crucially affect the optimal pricing of the premium version. Under the assumptions that the consumers' valuations for the advertised product follow a uniform distribution and that advertising has a classical signaling structure, we relate the optimal pricing strategy to the underlying degree distribution and the hazard rate of the random network. We derive close form expressions for the platform's profits in most prominent real-world social networks where online platforms operate. In the model, platforms that operate in large and relatively sparse networks have incentives to provide only the free version of the service—alongside with advertising—, whereas platforms serving more densely connected networks prefer to provide both versions of the service.

Keywords: Social networks, second-degree discrimination, advertising, degree distributions, hazard rate

JEL Classification: D83, D85, L1, M3

1 Introduction

Consider a group of consumers engaged in a social network that entails positive externalities over the consumption of a service offered by an online platform. Suppose that the platform has the option of selling a two-version bundle of the service that consists of: (1) a free version,

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which includes advertising about some unrelated consumption product, and (2) a premium version, which comes with no advertising and allows higher network externalities relative to the free version. In these situations, the revenue to the platform usually depends both on its sales of the premium version of the service and on the profitability of its advertising activity through the free version. Interestingly, there is a trade-off between both incentives since higher proportions of premium version adopters imply lower proportions of consumers who receive ads about the consumption product. The problem of the platform is then to choose an optimal second-degree discrimination policy, where consumers self-select themselves to adopt one version of the service or the other. Over the recent years, this type of discrimination schemes with advertising have become a prevalent practice by online platforms that operate over social networks.[1] This business model is known as *versioning* in the business literature.

In a companion paper, under the assumptions that (1) advertising rotates clockwise the inverse demand of the advertised product and (2) the platform receives a fixed portion of the revenue from the sales of the advertised product, Jimenez-Martinez (2018) analyzes the incentives of the platform to pursue versioning, its optimal pricing policy, and the welfare implications. The model proposed there considers that the social network is randomly generated and allows for a general class of demand functions for the advertised product. In such a framework, the results on optimal pricing, as well as the derived welfare implications, can be related to the degree distribution that underlines the social network and, in particular, to its hazard rate. Yet, given that it proposes a fairly general model—with respect to degree distributions, demand functions, and effects of advertising—, a caveat of Jimenez-Martinez (2018)'s analysis is that it does not allow one to derive closed form expressions neither for the platform's profit functions nor for the optimal pricing policy. In some cases, this makes it hard to map the general implications of the model to sharp descriptions of how real-world platforms operate over social networks. The goal of this paper is to obtain more detailed results from the model on versioning proposed by Jimenez-Martinez (2018), using more specific assumptions about demand functions and about the effects of advertising on demand. In particular, we assume that the distributions of the product's valuations that underline demand functions are

[1] Prominent platforms that offer either a two-version bundle of their services or only a free version with advertising include Google and Yahoo (web search engine), Facebook, Twitter, Instagram, and Snapchat (social interaction), Whatsapp, Skype, and Line (communication) AirBnB (accommodation search), Amazon (retail, product search, and market matching), Waze (traffic and route forecasting), BBC, CNN, The New York Times (news), The Weather Channel (weather forecasting), Yelp and Foursquare (review and rating), YouTube, Vimeo, Apple—through the Apple Music application—, and Spotify (entertaining and information), Strava, MapmyRun, Nike Training Club, and Azumio (exercise and health tracking), Box (second-hand trade), or Tinder (dating). Some of these platforms have recently switched between offering the two-version bundle and providing only the free version (e.g., YouTube, Amazon, Waze, CNN, and Kayak).

uniform and that advertising has a classical signaling structure. Under these further assumptions, the model is able to capture—with a fairly good degree of precision—empirical regularities of second-degree discrimination with advertising in social networks. In particular, the applications of Section 4 allow us to explore in detail the platform's profit functions for the two most prominent classes of degree distributions which, according to the available empirical evidence, generate most social networks: the power law pattern and the exponencial pattern.[2] Our paper can thus be of interest for practitioners in the area and for empirical analysis. According to the model, platforms that operate in large and relatively sparse networks—typically generated by power law distributions—have incentives to provide only the free version of the service, whereas platforms serving more densely connected networks—typically generated by exponential distributions—prefer to provide both versions of the service.

We consider that the social network is randomly generated. Random networks are usually best interpreted from a dynamic perspective where the *underlying degree distribution* determines how the average neighborhood size evolves over time. In our model, the role of the random network in the platform's optimal pricing is related to the degree distribution that generates the network and to its *hazard rate function*. On the one hand, the shape of the degree distribution directly determines the demands of both versions of the service. Other things equal, because of the difference in network externalities, consumers with relatively large neighborhoods (or that expect relatively large neighborhoods in the future, for dynamic interpretations of our model) prefer the premium version rather than the free one. As a consequence, the demand of the premium version is positively related with the probability that the network generates relatively large neighborhoods. In Proposition 1, we show that the shape of the hazard rate function plays a key role in the platform's optimal pricing decisions.

Given that our main goal is the relationship between the platform and the consumers, this paper abstracts from the plausible—in practice, very diverse and complex—relations that could exist between the platform and a different/separate company offering the advertised product. Using a reduced form, we model both companies as being perfectly integrated and acting as a single monopolist that offers both commodities, the service and the advertised

[2] Power law distributions are suggested to describe well the nodes on the Internet (Barabási and Albert (1999)), the number of links in websites (Clauset et al. (2009)), and the social commerce network through the Internet (Stephen and Toubia (2009)). Also, in perhaps the largest structural analysis conducted up to date, Ugander et al. (2011) conclude that the Facebook social network features decreasing hazard rates—though not necessarily adjusting to the pattern of a power law distribution. On the theoretical side, decreasing hazard rates are always generated by two ubiquitous models: the *preferencial attachment model* proposed by Barabási and Albert (1999) and the *network-based search model* suggested by Jackson and Rogers (2007). On the other hand, Rosas-Calals et al. (2007), Ghoshal and Barabási (2011), and Scholz (2015) provide empirical evidence which suggests that some social networks adjust well to the pattern of exponential degree distributions.

product.[3] Also, the analysis focuses on networks of consumers where all of them are in the first place interested in purchasing at least the free version of the service.

Our article is related to a prolific literature, initiated by Farrell and Saloner (1985) and Katz and Shapiro (1985), that explores the effects of network externalities on economic decisions and to the classical second-degree discrimination analysis of Mussa and Rosen (1978) and Maskin and Riley (1984), wherein a monopolist offers a menu of different qualities of a single product. Our notion of advertising builds upon the *informative advertising* approach pursued by Lewis and Sappington (1994) and Johnson and Myatt (2006). Specifically, we consider that advertising informs consumers about the product's characteristics and helps them improve their knowledge about their true underlying preferences for the product. Also, there is a profound connection between the effects of advertising that we consider and the structure of the endogenous information decision problem explored by Amir and Lazzati (2016) in the context of (common value) Bayesian supermodular games proposed by Van-Zandt and Vives (2007).[4] First, as in these games, consumers' valuations for the advertised product and advertising are complements in our model. Given this, because of the informative role of advertising that we assume, the inverse demand of the advertised product rotates clockwise. Although their problem has a different motivation, Amir and Lazzati (2016)'s assumption of an information structure that is convex in the supermodular order is conceptually similar to our demand rotation implications.

Our technical analysis relies on the random networks literature initiated with Erdös and Renyi (1959). In particular, our dynamic interpretation of random networks meets the assumptions of the canonical *configuration model* which was originally proposed by Bender and Canfield (1978) and used subsequently by a number of influential papers in the social networks area.[5]

Perhaps the paper most closely related to ours is Gramstad (2016). While Gramstad (2016) explores the role of the network structure in optimal pricing in an environment where con-

[3]While this assumption makes the analysis tractable, it captures natural situations where the platform's profits increase when the impact of advertising makes the sales of the advertised product more profitable. Furthermore, in many real-world cases it is actually the case that the platform and the company that sells the advertised product have some degree of integration (The Weather Channel and IBM, Nike Training Club and Nike, MapmyRun and Under Armour, Apple Music and Beats, or Amazon and Whole Foods Market and Zappos). Reportedly, Facebook has acquired 65 different businesses, Apple participates in 91 different companies, Google has acquired (either fully or partially) over 200 companies from 16 different countries. The products offered by the companies integrated with the companies that own the platform are often advertised by the corresponding platform.

[4]Within the organizational literature, Dessein et al. (2016) explore an attention choice problem with an analogous supermodular structure.

[5]See, e.g., Bollobás (2001), Newman et al. (2001), Jackson and Yariv (2007), Galeotti and Goyal (2009), and Fainmesser and Galeotti (2016)).

sumers choose between different versions of some product, as we do, there are, though, important differences between both papers. The main one is that the role of a mechanism such as advertising on the platform's profits is absent in her analysis. In addition, we consider that the size of the network externality depends only on the version chosen by the consumer himself and not on the versions chosen by her neighbors, whereas Gramstad (2016) requires consumers to choose the same version for enjoying the externalities. As a consequence, Gramstad (2016)'s analysis focuses on segmented markets along the network, whereas we consider a single market with different versions of the product.

The rest of the paper is organized as follows. Section 2 lays out the model and Section 3 describes the paper's main result (in Proposition 1). Section 4 applies the model to the classes of random networks which are generated by power law and exponential distributions, and Section 5 concludes. The proof of the main result in Proposition 1 is relegated to the Appendix.

2 The Model

There is unit mass of consumers, indexed by $i \in [0,1]$, embedded in a (complex) social network. A platform offers two unrelated commodities over the social network: (1) $z \geq 0$ units of a consumption product, which it produces at a marginal cost $c \in (0, 1/2)$,[6] and (2) any discrete quantities of a two-version online service bundle, which it produces at no cost. The network entails positive externalities for the consumption (only) of the service: the utility of each consumer from her consumption of the service increases with the number of her neighbors. The two-version service bundle consists of: (a) a *free version*, which is offered with advertising about the product, and (b) a *premium version*, which does not include advertising and, furthermore, allows consumers higher externalities relative to the free version. Throughout the paper, the notation $a = 0$ or $a = 1$ will be used to indicate, respectively, that advertising is either absent or present. Use $p \geq 0$ to denote the price of each unit of the consumption product and $q \geq 0$ to denote the price of the (premium version of the) service. By offering two different versions of the service, the platform is able to implement a particular form of second-degree discrimination policy where consumers decide which version they purchase. This, in turn, influences (only) free version adopters' opinions about the advertised product.

Each consumer has a unit demand for the service, and the expressions "purchase" or "adopt" a version of the service will be used exchangeably. Let z_i be consumer i's probability of purchasing the product and let x_i be the probability that consumer i adopts the premium version

[6]The production cost of the advertised product is naturally assumed to be lower than its prior expected valuation.

of the service. Each consumer is willing to pay up to ω for a unit of the product and up to θ for a unit of the service. The consumers' valuations for the product ω are independently drawn from a uniform distribution $U[0,1]$ and their valuations for the service θ are independently drawn from some interval $(\underline{\theta}, \overline{\theta})$, with $\underline{\theta} > 0$, according to some (common) distribution. The two commodities are totally unrelated and, therefore, the valuations ω and θ are assumed to be independent from each other. Since consumers pay no price for the free version, the assumption that $\underline{\theta} > 0$ allows us to restrict attention to situations where all consumers have in the first place a positive net interest in the service. In other words, by construction, the analysis focuses on groups of consumers where all of them are willing to adopt at least the free version of the service. This makes sense if we consider that such "minimally interested" consumers are indeed the ones who form the network targeted by the platform. Given this, we then explore the conditions under which some consumers are willing to "go further' and adopt the premium version rather than the free one.

Formally, the platform and the consumers are engaged in a game where the platform chooses a *price* $p \geq 0$ *for the consumption product* and a *price* $q \geq 0$ *for (the premium version of) the service*, whereas the consumers make simultaneous *consumption decisions* $(z_i, x_i)_{i \in [0,1]}$ about the service and the product. Price discrimination is not allowed for the advertised product so that all consumers face the common price p, regardless of the version of the service that they adopt.

2.1 Signaling Advertising

We consider that advertising is informative and, in particular, that it helps consumers to improve their knowledge of their own tastes for the product. Specifically, consumers are uncertain about their valuation ω of the product. Prior to their purchasing decisions, they do not observe ω but receive some (noisy) *public signal realization* $y \in (1/2, 1]$.[7] The signal realization y can be interpreted as the public observation of some posted information about the product quality (such as the one obtained from existing commercials, marketing samples, or from some other selling activities). Consumers use the information provided by the signal realization y to update their beliefs about their underlying valuations ω. The public signal realization y and the advertising level $a \in \{0,1\}$ influence the consumers' posterior valuations of the product

[7]The signal realization is naturally assumed to the higher than the expected prior of the advertised product's valuation.

according to a very simple signaling structure. In particular, the public signal yields either:

(1) the actual realization of the valuation $y = \omega$, with probability $\dfrac{1+a}{2}$, or

(2) some independent drawn from $U[0,1]$, with probability $\dfrac{1-a}{2}$.

Given this signaling structure, each consumer i obtains the posterior expectation of her valuation for the product by applying Bayes rule:[8]

$$E[\omega \mid y, a] = \frac{1+a}{2} y + \frac{1-a}{2}\left(\frac{1}{2}\right). \tag{1}$$

Although all consumers are able to use the signal realization y to revise their priors, the posteriors of those consumers that indeed receive advertising place more weight—relative to the consumers that do not observe advertising—on the signal realization y. Our simple specification entails that when advertising is absent ($a = 0$), consumers place weights according to a uniform probability on either the public signal realization y or their prior $1/2$ being their true underlying valuations, $E[\omega \mid y, 0] = (1/2)y + (1/2)(1/2)$. On the other hand, when advertising is present ($a = 1$), consumers are able obtain additional information and turn to assign probability one to the valuation $\omega = y$ and probability zero to the prior: consumers learn their true valuations, $E[\omega \mid y, 1] = y$.

This simple formulation builds on the settings proposed by Lewis and Sappington (1994) and Johnson and Myatt (2006) to analyze informative advertising. In particular, our specification of signaling ads induces the type of fairly general clockwise demand rotations which were proposed by Johnson and Myatt (2006) and subsequently used by Jimenez-Martinez (2018). To see this, let us use $P_a(z)$ to denote the inverse demand of the product if *all* consumers receive an advertising level $a \in \{0,1\}$ and suppose that the platform offers a quantity z of the product. Since the product's valuation is drawn from a $U[0,1]$ distribution, the platform sells the product to those consumers receiving a public signal greater than $y = 1 - z$. The platform must therefore set prices according to the linear rules:

$$\begin{aligned} P_0(z) &= (3/4) - (1/2)z; \\ P_1(z) &= 1 - z, \end{aligned} \tag{2}$$

which gives us the expressions for the induced inverse demands of the product, conditioned on each advertising level $a \in \{0,1\}$. Figure 1 displays the linear demands functions that thereby stem from the assumption of uniform valuations for the product, together the considered signaling structure of advertising. It can be easily verified that the induced inverse demand $P_a(z)$

[8] In the expression in (1), the unconditional prior is given by $E[\omega] = 1/2$ since $\omega \sim U[0,1]$.

rotates clockwise around the rotation point $(z^R, p^R) = (1/2, 1/2)$ as a moves from $a = 0$ to $a = 1$, exactly as required by Definition 1 of Johnson and Myatt (2006), and as considered in Assumption 1 of Jimenez-Martinez (2018).

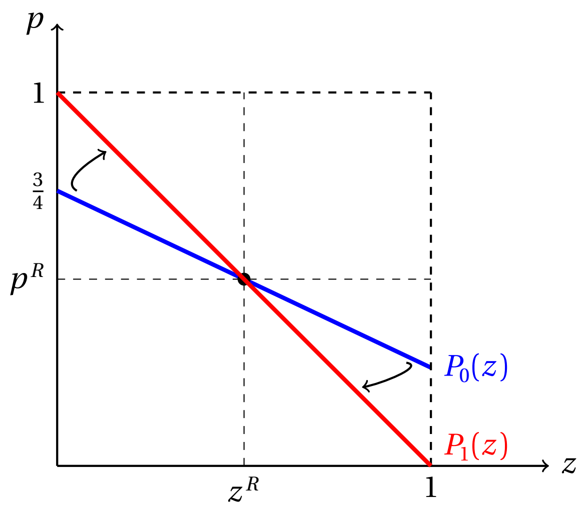

Figure 1: Rotation of Inverse Demand Induced by Adverting

2.2 The Social Network and Consumers' Preferences

The targeted social network is stochastic and exogenously given.[9] While considering a random network makes the analysis tractable, it allows for interesting dynamic interpretations of how neighborhoods' sizes evolve over time, which can formally be related to the optimal pricing policy of the platform.

The platform and the consumers are uncertain about the specific architecture of the network but they commonly know the stochastic process that generates it. There is a set $[\underline{n}, \overline{n}] \subseteq \mathbb{R}_+$ of possible neighborhood sizes, or *degrees in the social network*.[10] Let $n_i \in [\underline{n}, \overline{n}]$ be a possible *degree for consumer* i. The *degree distribution of the social network* is given by a twice continuously differentiable distribution $H(n)$, with strictly positive density $h(n)$, over the support $[\underline{n}, \overline{n}]$.[11]

[9]The links facilitated by the platform are included in the targeted social network but the network can in principle be a broader graph. Conceivably, the network can include also family, friendship, working relations links, or links provided by other platforms. Then, one would naturally consider that the platform cannot create or remove family, friendship or other links which are external to the platform's provision of its online service bundle.

[10]For some applications, one might consider $\underline{n} = 0$. Also, the set $[\underline{n}, \overline{n}]$ could be unbounded as well and, in particular, $\overline{n}$ could tend to infinity in some cases.

[11]In other words, the degree of a fraction $H(n) = \int_{\underline{n}}^{n} h(m)dm$ of consumers does not exceed n.

Let $x(q)$ be the fraction of premium version adopters at price q, conditioned on the degree distribution $H(n)$. Since each consumer is assumed to adopt one of the two versions of the service (which is ensured by the assumption $\underline{\theta} > 0$), it follows that the average consumption of the service of a consumer's neighbor equals one.[12] Therefore, in a quite convenient way, the number n_i captures in our setup the average consumption of the service of agent i's neighbors, conditioned on consumer i having degree n_i.

The network allows consumers to interact locally with respect to their consumptions (only) of the service. In particular, the consumption of the service exhibits a local (positive) network effect: a consumer's utility from—any version of—the service increases as her neighbors increase their consumptions[13]—regardless of the version of the service that the neighbors adopt—. Specifically, a consumer's utility raises by an amount of 1, when he adopts the free version of the service, or by an amount $1+\beta$, where $\beta > 0$, when he adopts the premium version, for each unit of the service consumed by her neighbors. The term β thus describes the presence of an *externality premium*, which seems to be the consistent with most real-world online services where the premium version allows consumers to enjoy the network externalities to a greater extent, compared to the free version.

Finally, we consider that consumers process their information about their neighbors according the *degree independence assumption*. This assumption requires that each member of the network regards her links as independently chosen from the random network[14] and guarantees that her degree is the only relevant information about the network that each consumer needs to consider.

Given all the considerations above, the expected utility of a consumer i, conditioned on

[12]Formally, under the assumptions of the *configuration model*, the expected consumption of some version of the service of a consumer i's neighbor can be computed as

$$\frac{\int_{\underline{n}}^{\overline{n}} m h_s(m)[\mathbb{P}(z_i=1 \,|\, n_i=m)+\mathbb{P}(z_i=0 \,|\, n_i=m)]dm}{\int_{\underline{n}}^{\overline{n}} n h_s(n) dn} = 1$$

since, given the assumption $\underline{\theta} > 0$, we directly obtain $\mathbb{P}(z_i = 1 \,|\, n_i = m) + \mathbb{P}(z_i = 0 \,|\, n_i = m) = 1$ for each degree $m \in [\overline{n}, \underline{n}]$.

[13]In practice, such externalities take the form of benefits from being able to interact with a higher number of people (e.g. social interaction, second-hand trading, exercise tracking, or dating services), informational gains (e.g., weather forecast, traffic monitoring, news services, or review and rating services), or collaborative gains (e.g., online gaming or collaborative projects).

[14]This is a very common assumption in the literature on random networks and has been used, among others, by Jackson and Yariv (2007), Galeotti et al. (2010), Fainmesser and Galeotti (2016), and Shin (2016).

having degree n_i, is specified as

$$u(z_i, x_i \mid n_i) \equiv \underbrace{z_i(\omega - p)}_{\text{product}} + \underbrace{x_i[\theta - q + (1+\beta)n_i]}_{\text{premium version of the service}} + \underbrace{(1-x_i)[\theta + n_i]}_{\text{free version of the service}} . \tag{3}$$

From the preference specification in (3), it follows that the fraction of free version adopters at price q, conditional on the degree distribution $H(n)$, is given by

$$\begin{aligned} 1 - x(q) &= \mathbb{P}(x_i = 0) = \mathbb{P}(\theta - q + (1+\beta)n_i \le \theta + n_i) \\ &= \mathbb{P}\big(n_i \le q/\beta\big) = H\left(q/\beta\right). \end{aligned} \tag{4}$$

For price q, let $n(q) \equiv q/\beta$ be the cutoff degree such that $x_i = 0$ if $n_i \le n(q)$ whereas $x_i = 1$ if $n_i > n(q)$. From the expression in (4) above, we observe that the fraction of consumers that purchase the premium version naturally decreases in its price q. The sensitivity of the demand $x(q)$ of the premium version with respect to its price depends on the random process that generates the network and on the externality premium. In particular, we have $x'(q) = -\big(1/\beta\big)\, h(n(q))$. Finally, note that all consumers purchase the free version of the service (i.e., $x(q) = 0$) when its price satisfies $q \ge \overline{q}$, where the upper bound $\overline{q}$ on the service price is given by $\overline{q} \equiv \beta \overline{n}$. Since we are interested in restricting attention to optimal policies such that $q \in (\underline{\theta}, \overline{\theta})$, we will accordingly consider throughout the analysis that $\overline{\theta} \ge \overline{q} \equiv \beta \overline{n}$.

Hazard rate analysis turns out very useful in the proposed setup to capture key features of how the random network evolves in dynamic interpretations. Also, the main insights on the platform's optimal pricing can be conveniently related to the hazard rate of the degree distribution. The *hazard rate function* of the random network with distribution degree $H(n)$ is the function on $[\underline{n}, \overline{n}]$ defined as

$$r(n) \equiv \frac{h(n)}{1 - H(n)}. \tag{5}$$

For dynamic interpretations where the network evolves along several periods, the function $r(n)$ gives us the probability that a randomly selected consumer has approximately n neighbors in a subsequent period,[15] conditioned on her current neighborhood size being no less than n. Finally, note that the hazard rate of the degree distribution underlying the random network gives us key information about the sensitivity of the demand for the premium version of the service. In particular, we have $r(n(q)) = -\beta\, x'(q)/x(q)$.

[15]Formally, for the continuous distribution case, $r(n)$ is the probability that the number of neighbors of a randomly selected consumer lies in the interval $(n - \varepsilon, n + \varepsilon)$, for $\varepsilon > 0$ sufficiently small.

3 Main Result on Optimal Pricing

Let $z_a(p)$ and $\pi_a(p)$ denote, respectively, the demand of the product and the profits to the platform from the product's sales, at price p, conditioned on the advertising level a. Recall that the platform does not price discriminate with respect to the consumption product and, therefore, sets a common price p for all consumers, regardless of the version of the service that they adopt. In the absence of advertising ($a = 0$), we then derive (from the expression in (2) for the inverse demand): $z_0(p) = 3/2 - 2p$ for $p \leq 3/4$, which in turn leads to the profits

$$\pi_0(p) = \begin{cases} (p-c)\left[\frac{3}{2} - 2p\right] & \text{if} \quad p \in (0, \frac{3}{4}] \\ 0 \quad \text{if} \quad p \in (\frac{3}{4}, 1]. \end{cases} \tag{6}$$

In the presence of adverting ($a = 1$), we then derive (from the expression in (2)): $z_1(p) = 1 - p$ for $p \leq 1$, which leads to the profits

$$\pi_1(p) = (p-c)[1-p] \quad \text{if} \quad p \in (0, 1]. \tag{7}$$

The platform makes its optimal pricing choices following backwards induction through a two-stage process. Given a degree distribution $H(n)$ that underlines the random network and a price $q \in (\underline{\theta}, \overline{\theta})$ of the service, the platform's goal is first to set a price $p^* \in (0, 1)$ for the product so as to maximize the profits from the product's sales

$$\pi(p, q) \equiv x(q)\pi_0(p) + [1 - x(q)]\pi_1(p). \tag{8}$$

Since advertising affects (some of) the consumers' willingness to pay for the product, the profits from the product's sales naturally depends on the fraction $x(q)$ of premium version adopters. Then, given an optimal choice $p^* \in (0, 1)$ for the price of the product, the platform wishes to choose a price $q^* \in (\underline{\theta}, \overline{\theta})$ for the service so as to maximize its overall profits

$$\Pi(p^*, q) \equiv q\, x(q) + \pi(p^*, q). \tag{9}$$

Formally, the optimal choices $((p^*, q^*), (z_i^*, x_i^*)_{i \in [0,1]})$ correspond to a *Nash equilibrium* of the described game where the platform chooses the prices of the product and the service and consumers choose their purchases of the product—which determines the demand $z(p^*)$ of the advertised product—and self-select themselves to adopt one version or the other of the service—which determines the demand $x(q^*)$ of the premium version of the service—. Existence of equilibrium is guaranteed since the profits specified in (8) and (9) are continuous functions on compact convex sets.

Proposition 1 provides the key necessary conditions that must satisfy the platform's optimal pricing strategy for interior prices in our model.

Proposition 1. *Consider a random social network with degree distribution $H(n)$ and hazard rate function $r(n)$. Suppose that the platform optimally chooses interior prices $p^* \in (0,1)$, for the advertised product, and $q^* \in (\underline{\theta}, \overline{\theta})$, for the service, with an equilibrium fraction of premium version adopters $x(q^*) \in (0,1)$. Then, such optimal prices must satisfy:*

$$p^* = \frac{1}{2}\left(\left[\frac{1+x(q^*)/2}{1+2x(q^*)}\right]+c\right) \tag{10}$$

and

$$q^* + \Phi(q^*) = \frac{\beta}{r(n(q^*))}, \tag{11}$$

where $\Phi(q)$ is a function that decreases strictly in q if $x(q) > 1/7$ (equivalently, if $H(n(q)) < 6/7$) and increases strictly if $x(q) < 1/7$ (equivalently, if $H(n(q)) > 6/7$).

A direct corollary to Proposition 1 is that the (interior) optimal prices of the advertised product and the service always move in the same direction. In particular, it follows from the expression provided by (10) that

$$\frac{dp^*}{dq^*} = -\frac{2}{3\left[1+2x(q^*)\right]^2}\left(-\frac{1}{\beta}\right)h(n(q^*)) > 0.$$

In other words, the best-reply that relates the platform's optimal choices q^* and p^* has a positive slope.

Since the function $\Phi(q)$ identified in the equilibrium condition (11) of Proposition 1 is not monotone in q, uniqueness of equilibrium cannot be guaranteed in general even when the hazard rate $r(n(q))$ is monotone in q. Nevertheless in our two main applications, which capture most real-world social networks targeted by online platforms—according to the available empirical evidence—, equilibrium is unique.

4 Main Applications

This section presents two applications of the model to two prominent degree distributions which underline most social networks targeted by online platforms. Both empirical evidence and theoretical models suggest that most (complex) social networks are fairly well captured by either scale-free/power law patterns or by exponential degree distributions. In particular, the empirical evidence obtained by Barabási and Albert (1999), Clauset et al. (2009), Stephen and Toubia (2009), Ugander et al. (2011), together with the theoretical micro-foundations recently provided by Shin (2016), suggest that large and relatively sparse social networks—usually, Internet-based—adjust well to the pattern of power law distributions. On the other hand, other empirical findings (Rosas-Calals et al. (2007), Ghoshal and Barabási (2011), and Scholz (2015)) point in the direction that smaller and more connected networks—usually, Application-based—adjust relatively better to exponential distributions.

For the proposed model, the proof of Proposition 1 shows that the platform's profits for each (interior) optimal price p^* of the advertised product are given by

$$\Pi(p^*, q) = qx + \frac{1}{4}\big[\eta(x) - c\big]\big[1 + [1 - \eta(x) - c][1 + x]\big], \tag{12}$$

where x stands for short-hand notation of the fraction of premium version adopters at price q (i.e., $x = x(q)$), and $\eta(x)$ denotes the function $\eta(x) \equiv [1 + x/2]/[1 + 2x]$. The expression above for the platform's profits $\Pi(p^*, q)$ will be used in our two key applications. Also, the proof of Proposition 1 specifies the function $\Phi(q)$ identified in the proposition, which will be used in the applications to compute optimal prices for the service, by using the composite function $\phi(x)$ specified as

$$\phi(x) \equiv \frac{1}{4}\Big[\eta'(x)\Big[1 + [1 + x][1 - 2\eta(x)]\Big] + [\eta(x) - c][1 - \eta(x) - c]\Big],$$

with $\phi(x(q)) \equiv \Phi(q)$. In addition, we will consider in both applications a cost parameter $c = 1/4$ and a value $\beta = 1$ for the externality premium, which implies $n(q) = q$.

4.1 Power Law Degree Distribution

Empirical evidence suggests that most real-world online and Internet-based social networks are *scale-free*. Thus, if we consider that such networks are randomly generated, then the corresponding degree distribution must follow a *power law*. Also, power law degree distributions are particularly suitable to model the formation of networks that follow a preferential attachment pattern. For a degree support $[\underline{n}, +\infty)$, with $\underline{n} > 0$ a *power law degree distribution with parameter* $\sigma > 1$, is given by

$$H(n) = 1 - \underline{n}^{\sigma-1} n^{-(\sigma-1)}.$$

Most empirical estimates propose values for the parameter σ that lie in the interval $(2, 3)$. The corresponding hazard rate function is $r(n) = (\sigma - 1)/n$, which decreases in n. Let us consider $\sigma = 5/2$ and $\underline{n} = 1$. Then, the demand of the premium version of the service is given by $x(q) = q^{-3/2}$ so that $\eta(x) = [2 + q^{-3/2}]/[2 + 4q^{-3/2}]$. From the expression provided in (12), we obtain that the platform's profit function $\Pi(p^*, q)$ takes the form

$$\Pi(p^*, q) = q^{-1/2} + \frac{1}{4}\left(\frac{2 + q^{-3/2}}{2 + 4q^{-3/2}} - \frac{1}{4}\right)\left(1 + \left[\frac{3}{4} - \frac{2 + q^{-3/2}}{2 + 4q^{-3/2}}\right]\left[1 + q^{-3/2}\right]\right).$$

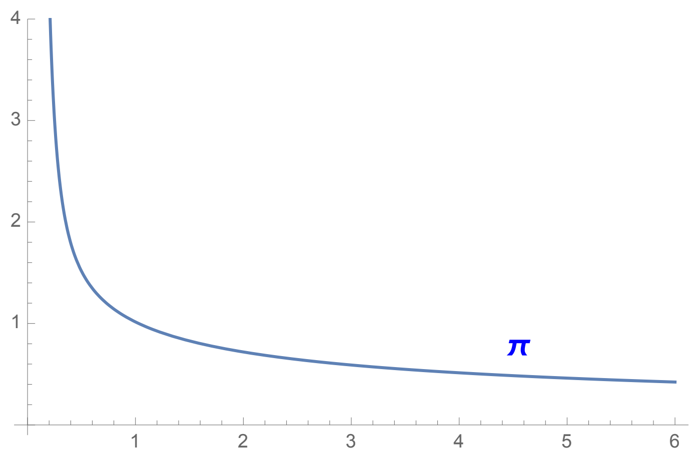

Figure 2: Platform's profits for the power law degree distribution.

The graph of such a profit function is depicted in Figure 2.

The necessary condition provided by Proposition 1 in (11) is not satisfied in this application. We observe that the platform's optimal choice entails a corner solution. Specifically, the platform's profits are always decreasing in the price of the (premium version of) the service. It then follows that providing the premium version is not profitable and the platform chooses optimally to serve only the free version of the service.

4.2 Exponential Degree Distribution

A typical degree distribution with constant hazard rate function is the *exponential degree distribution*. At a theoretical level, exponential degree distributions are often used to capture the formation of links according to uniform randomness in growing random networks.[16] For a degree support $[0,+\infty)$, an *exponential degree distribution with parameter* $\sigma > 0$ is given by

$$H(n) = 1 - e^{-n/\sigma}.$$

The corresponding hazard rate function is $r(n) = 1/\sigma$. Let us consider $\sigma = 1$. Then, the demand of the premium version of the service is given by $x(q) = e^{-q}$ so that $\eta(x) = [2+e^{-q}]/[2+4e^{-q}]$. From the expression provided in (12), we obtain that the platform's profit function $\Pi(p^*, q)$ takes the form

$$\Pi(p^*, q) = q\,e^{-q} + \frac{1}{4}\left(\frac{2+e^{-q}}{2+4e^{-q}} - \frac{1}{4}\right)\left(1 + \left[\frac{3}{4} - \frac{2+e^{-q}}{2+4e^{-q}}\right]\left[1+e^{-q}\right]\right).$$

The graph of such a profit function is depicted in Figure 3.

[16] See, e.g., Jackson (2008), Chapter 5, for an insightful description of the use of the exponential degree distribution in growing random networks.

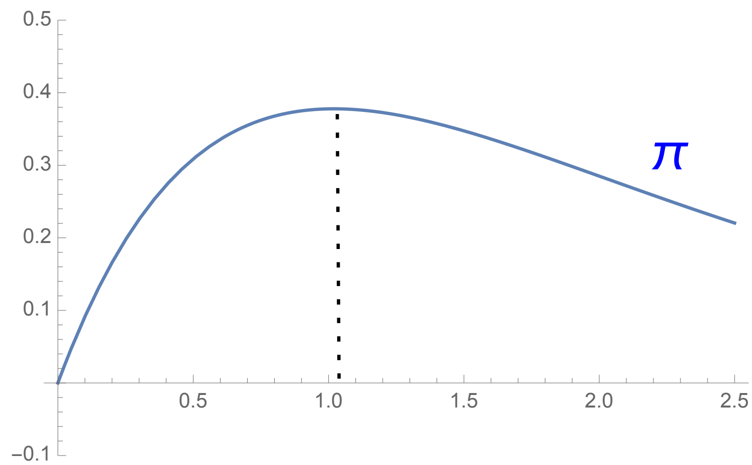

Figure 3: Platform's profits for the exponential degree distribution.

The necessary condition provided by Proposition 1 in (11) yields the interior optimal price $q^* \approx 1.056$. In this application, the platform thus optimally chooses to pursue versioning by offering both versions of the service.

4.3 Comparative Implications

The comparative implications that stem from the two applications explored above are in consonance with the results obtained by Jimenez-Martinez (2018) in his more general setup. Interestingly, the two applications of this section allow us to derive closed forms for the platform's profits—conditioned on its optimal choice for the price of the advertised product—and, therefore, to appreciate in detail the incentives for pursuing versioning. We have obtained that platforms operating in social networks that feature power law patterns have strong incentives to offer only the free version of their services, whereas social networks governed by exponential patterns do provide incentives for platforms to rely on both versions of their services. Therefore, by resorting to the available empirical findings and theoretical developments (Barabási and Albert (1999), Clauset et al. (2009), Stephen and Toubia (2009), Ugander et al. (2011), Barabási and Albert (1999), Rosas-Calals et al. (2007), Ghoshal and Barabási (2011), Scholz (2015), and Shin (2016)) our model's implications lead to that the platform has more incentives to rely only on its advertising activity in large and sparse social networks, compared to the cases of smaller and more connected networks. To illustrate further these messages, Figure 4 provides the density measures[17] of specific real-world social networks where some

[17]The density measure of a network specifies the proportion of possible links that are actually present in the network. Algebraically, it is given by the coefficients $l/n(n-1)$, for directed networks, and $2l/n(n-1)$, for undi-

Platform	Size (number of nodes) of the Social Network	Density Measure (%) of the Social Network
Google	875,713	0.00066
YouTube	1,134,890	0.00046
Amazon	334,863	0.00165
Twitter	81,306	0.0267
Google+	107,614	0.118

Figure 4: Density Measures of Social Networks

prominent platforms operate. Such density measures have been computed using the data provided by the Stanford Large Network Dataset Collection (Leskovec and Krevl (2014)).

We observe that the networks served by these prominent platforms are large and relatively disperse. Notably, Google, Facebook, and Twitter only serve the free version with advertising. As for the Application-based social networks which are targeted by exercise and health tracking platforms, as far as we know, there is no empirical evidence about their density measures. However, if we consider social networks that are restricted geographically, then naturally most of the service users are connected with other local users who are using the application.[18]

5 Concluding Remarks

The growth of online and assorted technologies has made it possible for platforms that operate in social networks to implement novel forms of discrimination strategies which rely heavily on advertising. These new possibilities have motivated an array of important questions about how the dynamic evolution of social networks interacts with advertising and pricing strategies. In this article, by restricting attention to uniform distributions of consumers' valuations of the advertised product and by considering a signaling role of advertising, we have investigated further the implications of the model on second-degree discrimination proposed, in a companion paper, by Jimenez-Martinez (2018).

There are different directions in which our model could be extended. First, we have considered the case of monopoly and it would be very interesting to explore the consequences of

rected networks, where l denotes the number of links present in the network and n indicates the number of nodes in the network.

[18]Casual observation suggests that this is the case for platforms such as Nike Training Club, Strava, or MapmyRun. Such local networks are certainly much smaller than the networks where Google or Twitter operate and, conceivably, each user can access to the information posted by most of their local users.

having several platforms competing in a common social network. Secondly, our framework assumes that the only factor that determines the size of the network externalities is the degree of the nodes. Thus, we have not incorporated important features present in real-world social networks (such as clustering or assortativity) which, in practice, could influence the consumption externalities. For example, the externality could be larger when common neighbors are shared or when the links are formed between consumers with some similar characteristics. Finally, our analysis has ignored the fact that the links could be used to transmit information about the characteristics of the advertised product. This information transmission would interact with the information distributed through formal advertising. In particular, for review and rating services, including this possibility into the analysis seems very important.

The specific assumptions used in the model have nonetheless allowed us to obtain useful close form expressions for the platform's profit functions in relevant social networks. The framework explored here is widely applicable in the dimension of the degree distribution, and it seems to capture empirical observations about how real-world platforms operate in their provision of online services through social networks.

Appendix

Proof of Proposition 1. Consider a random network with degree distribution $H(n)$ and hazard rate function $r(n)$. Using the expressions derived in (6)–(9) for the platform's profits, it follows that

$$\Pi(p,q) = \begin{cases} q\,x(q) + (p-c)\big[x(q)\big(\frac{3}{2}-2p\big)+[1-x(q)]\big(1-p\big)\big] & \text{if } p\in[0,\frac{3}{4}] \\ (p-c)\big[1-p\big] & \text{if } p\in(\frac{3}{4},1]. \end{cases}$$

Take given a price $q\in(\underline{\theta},\overline{\theta})$ of the service. Application of the first-order conditions for interior solutions $\partial\Pi(p,q)/\partial p = 0$ to both steps of the profit function above yields:

(a) for candidates to interior maximum $p^*\in\big(0,\frac{3}{4}\big)$, we have

$$x(q)\big(3/2-2p^*\big)+[1-x(q)]\big(1-p^*\big)-(p^*-c)\big[1+x(q)\big]=0 \Leftrightarrow p^* = \frac{\Big[\frac{1+x(q)/2}{1+2x(q)}\Big]+c}{2};$$

(b) for candidates to interior maximum $p^{**}\in\big(\frac{3}{4},1\big)$, we have

$$1-p^{**}-(p^{**}-c)=0 \Leftrightarrow p^{**} = \frac{1+c}{2}.$$

For $x\in[0,1]$, let us specify the function $\eta(x)\equiv[1+x/2]/[1+2x]$, where x is used as short-hand notation for the proportion of premium version adopters at price q, i.e., $x\equiv x(q)$. Observe that the candidate p^* for interior optimum obtained above can be rewritten as $p^*=[\eta(x)+c]/2$ for $x\in[0,1]$ and that p^* tends to $p^{**}=(1+c)/2$ as x tends to zero (that is, as all consumers become free version adopters). Therefore, using the function $\eta(x)$, any candidate to interior optimum can be described by $p^*=[\eta(x)+c]/2$ for $x\in[0,1]$. Note that $\eta'(x) = -3/2(1+2x)^2 < 0$, so that $\eta(x)$ always decreases in $x\in[0,1]$, with $\eta(0)=1$ and $\eta(1)=1/2$. Thus, $\eta(x)\in[1/2,1]$ for each $x\in[0,1]$. Also, using the function $\eta(x)$ together with the expressions for candidates to interior optima derived above, it can be verified that while $0<c<3/2-\eta(x)$ is required for $p^*\in(0,3/4)$ to be an interior optimum, $c>1/2$ is necessary for $p^{**}\in(3/4,1)$ to be an interior optimum. Therefore, since the production cost is assumed to satisfy $0<c<1/2$, we obtain that any interior optimum p^* must satisfy $p^*=[\eta(x)+c]/2$, with $x\in(0,1)$ and $\eta(x)\in(1/2,1)$. In addition, the second-order condition for the price choice p is automatically satisfied since $\Pi(p,q)$ is strictly convex in p. It follows that any interior optimal choice by the platform implies that it pursues versioning by offering both versions of its service, with $x(q^*)\in(0,1)$ and $(1/2+c)/2<p^*<(1+c)/2$. Since $c\in(0,1/2)$, it follows that if p^* is an interior optimal choice by the platform for the advertised product, then $p^*\in(1/4,3/4)$.

By plugging the expression $p^* = [\eta(x)+c]/2$ for the interior optimal price of the advertised product into the platform's profit function, it follows that

$$\Pi(p^*,q) = q\,x(q) + \frac{1}{4}\big[\eta(x)-c\big]\big[1+[1-\eta(x)-c][1+x]\big]. \tag{13}$$

Given the optimal choice $p^* \in (1/4, 3/4)$ for the advertised product, the first-order condition for interior solutions with respect to the price $q \in (\underline{\theta}, \overline{\theta})$ of the service then leads to

$$\begin{aligned}\frac{\partial \Pi(p^*,q^*)}{\partial q} =& x(q^*) + q^* x'(q^*) + \frac{\eta'(x)x'(q^*)}{4}\big[1+[1-\eta(x)-c][1+x]\big] \\ &+ \frac{\eta(x)-c}{4}\big[-\eta'(x)[x+1]+[1-\eta(x)-c]\big]x'(q^*) = 0.\end{aligned} \tag{14}$$

Since the density $h(n)$ is strictly positive over the degree support, we have that

$$x'(q^*) = -(1/\beta)h(n(q^*)) \neq 0.$$

Hence, by dividing over $x'(q^*)$ the expression obtained in (14) above, we derive the necessary condition

$$\begin{aligned}&\frac{x(q^*)}{x'(q^*)} + q^* + \frac{\eta'(x)}{4}\big[1+[1-\eta(x)-c][1+x]\big] \\ &\qquad + \frac{\eta(x)-c}{4}\big[-\eta'(x)[x+1]+[1-\eta(x)-c]\big] = 0.\end{aligned}$$

In turn, by rearranging the expression above, we obtain

$$q^* + \frac{1}{4}\Big[\eta'(x)\big[1+[1+x][1-2\eta(x)]\big]+[\eta(x)-c][1-\eta(x)-c]\Big] = -\frac{x(q^*)}{x'(q^*)}. \tag{15}$$

Let $\Phi(q)$ be the composite function specified as

$$\Phi(q) = \phi((x(q)) \equiv \frac{1}{4}\Big[\eta'(x)\big[1+[1+x][1-2\eta(x)]\big]+[\eta(x)-c][1-\eta(x)-c]\Big].$$

Application of the chain rule yields $\Phi'(q) = -(1/\beta)h(n(q))\phi'(x)$. Furthermore, by computing the required derivative, we obtain

$$\phi'(x) = \frac{1}{4}\Big[\eta''(x)\big[1+[1+x][1-2\eta(x)]\big]+\eta'(x)\big[2(1-2\eta(x))-2\eta'(x)[1+x]\big]\Big].$$

Using the specification of the function $\eta(x)$, we can derive the useful identities:

$$\begin{aligned}&\eta'(x) = -\frac{3}{2(1+2x)^2},\ \eta''(x) = \frac{6}{(1+2x)^3},\\ &1-2\eta(x) = \frac{x-1}{1+2x},\ 1+[1+x][1-2\eta(x)] = \frac{x(2+x)}{1+2x}.\end{aligned}$$

By plugging such calculations into the expression of the derivative $\phi'(x)$ above, it follows that:

$$\phi'(x(q)) = \frac{3\big(7x(q)-1\big)}{8\big(1+2x(q)\big)^4} > 0 \quad \Leftrightarrow \quad x(q) > 1/7.$$

Furthermore, it follows that:

$$\Phi'(q) = -\frac{h(n(q))}{\beta} \cdot \frac{3\big(6-7H(n(q))\big)}{8\big(3-2H(n(q))\big)^4} < 0 \quad \Leftrightarrow \quad H(n(q)) < 6/7 \text{ or, equivalently, } x(q) > 1/7.$$

Finally, by combining the implication that $r(n(q)) = -\beta\, x'(q)/x(q)$ and the condition derived in (15), we obtain the necessary condition for the platform's interior optimal choice:

$$q^* + \Phi(q^*) = \frac{\beta}{r(n(q^*))},$$

as stated. ∎

References

Amir, R. and N. Lazzati (2016), "Endogenous information acquisition in bayesian games with strategic complementarities." *Journal of Economic Theory*, 163, 684–698.

Barabási, A.-L. and R. Albert (1999), "Emergence of scaling in random networks." *Science*, 286, 509–512.

Bender, E. A. and E. R. Canfield (1978), "The asymptotic number of labelled graphs with given degree sequences." *Journal of Combinatorial Theory*, 24, 296–307.

Bollobás, B. (2001), *Random Graphs*, second edition. Cambridge University Press.

Clauset, A., C. Rohilla-Shalizi, and M. E. J. Newman (2009), "Power law distributions in empirical data." *Siam Review*, 51, 661–703.

Dessein, W., A. Galeotti, and T. Santos (2016), "Rational inattention and organizatonial focus." *American Economic Review*, 106, 1522–1536.

Erdös, P. and A. Renyi (1959), "On random graphs i." *Publicationes Mathematicae*, 6, 290–297.

Fainmesser, I. P. and A. Galeotti (2016), "Pricing network effects." *Review of Economic Studies*, 83, 165–198.

Farrell, J. and G. Saloner (1985), "Standardization, compatibility, and innovation." *The RAND Journal of Economics*, 16, 70–83.

Galeotti, A. and S. Goyal (2009), "Influencing the influencers: A theory of strategic diffusion." *RAND Journal of Economics*, 40, 509–532.

Galeotti, A., S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv (2010), "Network games." *Review of Economic Studies*, 77.

Ghoshal, G. and A.-L. Barabási (2011), "Ranking stability and super-stable nodes in complex networks." *Nature Communications*, 2.

Gramstad, A. R. (2016), "Nonlinear pricing with local network effects." Unpublished Manuscript.

Jackson, M. O. (2008), *Social and Economic Networks*. Princeton University Press.

Jackson, M. O. and B. W. Rogers (2007), "Meeting strangers and friends of friends: How random are social networks?" *American Economic Review*, 97, 890–915.

Jackson, M. O. and L. Yariv (2007), "Diffusion of behaviour and equilibrium properties in network games." *American Economic Review*, 97, 92–98.

Jimenez-Martinez, A. (2018), "Discrimination through versioning with advertising in social networks." *Economic Theory*, Forthcoming.

Johnson, J. P. and D. P. Myatt (2006), "On the simple economics of advertising, marketing, and product design." *American Economic Review*, 93, 756–784.

Katz, M. L. and C. Shapiro (1985), "Network externalities, competition, and compatibility." *American Economic Review*, 75, 424–440.

Leskovec, J. and A. Krevl (2014), "Snap datasets: Stanford large network dataset collection." http://snap.stanford.edu/data.

Lewis, T. R. and D. E. M. Sappington (1994), "Supplying information to facilitate price discrimination." *International Economic Review*, 35, 309–327.

Maskin, E. and J. Riley (1984), "Monopoly with incomplete information." *The RAND Journal of Economics*, 15, 171–196.

Mussa, M. and S. Rosen (1978), "Monopoly and product quality." *Journal of Economic Theory*, 18, 301–317.

Newman, M. E. J., S. H. Strogatz, and D. J. Watts (2001), "Random graphs with arbitrary degree distributions and their applications." *Physical Review*, E 64: 026118.

Rosas-Calals, M., S. Valverde, and R. V. Solé (2007), "Topological vulnerability of the european power grid under error and attacks." *International Journal of Bifurcation and Chaos*, 17, 2465–2475.

Scholz, M. (2015), "Node similarity as a basic principle behind connectivity in complex networks." *Journal of Data Mining and Digital Humanities*, http://jdmdh.episciences.org.

Shin, E. (2016), "Social network formation and strategic interaction in large networks." Unpublished Manuscript.

Stephen, A. T. and A. Toubia (2009), "Explaining the power law degree distribution in a social commerce network." *Social Networks*, 31, 262–270.

Ugander, J., B. Karrer, L. Backstrom, and C. Marlow (2011), "The anatomy of the facebook social graph." arXiv.

Van-Zandt, T. and X. Vives (2007), "Monotone equilibria in bayesian games of strategic complementarities." *Journal of Economic Theory*, 137, 339–360.