Making Friends: the Role of Assortative Interests and Capacity Constraints*

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Abstract

We study friendship networks under the assumption that agents are constrained in their efforts to build up links. Using a fairly general model, we investigate the relation between the prevailing (exogenous) assortative interests and the (endogenous) homophilic degrees of friendship patterns, and the welfare implications for such patterns. For intermediate assortative interests, extreme forms of homophilic (or heterophilic) patterns may coexist with more moderate forms. Under a very simple technology of link formation, the presence of capacity constraints leads to an interesting mechanism that makes certain amounts of heterophily (resp., homophily) necessary for extreme forms of homophilic (resp., heterophilic) patterns to be stable. Efficiency requires a form of common aggregate qualities of connections across all agents within each different population group. Under high (resp., low) assortative interests, some particular forms of only extreme homophilic (resp., heterophilic) patterns are simultaneously stable and efficient. For intermediate assortative interests, we identify a class of friendship networks that feature intermediate levels of homophily, and for which stability and efficiency are compatible.

Keywords: Friendship Networks, Assortative Interests, Homophily, Heterophily, Diversity, Integration

JEL Classification: A12, A14, D01, D71, D85, J15, Z13

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1 Introduction

Friendship relationships are central for individuals to (i) socialize and (ii) collaborate by complementing their abilities and information. The available literature suggests that individuals prefer to be friends with similar individuals in situations where the socialization motivation prevails.[1] On the other hand, when the main motivation is to collaborate, individuals prefer being friends with dissimilar individuals.[2] Lazarsfeld and Merton (1954) coined the term *homophily* to describe an observed tendency of people to lean towards others with similar characteristics, and the term *heterophily* to describe relations between individuals that differ in a certain characteristic.

But then, how do observed homophilic friendship patterns relate to the inner motivations to make friends? What happens to observed patterns if one shifts from environments with prevailing socialization motivations to others with the type of collaboration motivations? Are there other factors influencing observed homophilic patterns in friendship relations? We study how exogenous preferences for mating more similar (or dissimilar) individuals shape observed patterns of homophily. To fix notions, we will use the term *assortative* (resp., *disassortative*) *interest* to capture an exogenous taste according to which agents prefer to connect with similar (resp., dissimilar) individuals. We will then say "more or less *homophilic*" (resp., *heterophilic*) to describe how friendship relations among similar (resp., dissimilar) agents arise (endogenously) as stable.

Besides assortative tastes, we are interested in incorporating another key factor that influences the features of friendship patterns. Although individuals wish to make as many friends as possible and to build good-quality relations, they are critically constrained in their resources (e.g., time, information-load limits) to build friendship relations. *Capacity constraints* condition dramatically friendship connections.[3]

We propose a fairly general model where people form links that give rise to *friendship*

[1] Numerous arguments (e.g., self-identity concerns, risk-sharing measures, conflict prevention, and evolutionary selection) have been put forward to justify tastes that could lead to the documented prevalence of friendship relations between people with similar characteristics (Lazarsfeld and Merton, 1954; Felmlee et al., 1990; Mehra et al., 1998; Christakis and Fowler, 2014; McPherson et al., 2001; Heaton, 2002).

[2] The presence of complementaries (Newman, 2001; Moody, 2004; Guimera et al., 2005; Davis et al., 2003; Watts, 1999; Uzzi, 2008) have been crucially proposed to justify tastes that could explain friendship relations between people with different characteristics, documented mainly in the realm of scientific collaboration (Page, 2007) and of relations within organizations (Casciaro and Lobo, 2008).

[3] For instance, using data of neocortex volume of primates, Dunbar (1992a) suggests that our available neocortical neurons limit our information-processing capacities so as to restrict the number of relationships that we can monitor simultaneously. Also, using data from wild populations of baboons, Dunbar (1992b) argues that time constraints condition crucially the ability of individuals to form friendship connections. Social anthropologists commonly accept these type of empirical findings as the basis of the famous *Dunbar number* proposal, which places limits on the observed sizes of social groups and networks. For further evidence of the role played by time constraints see, e.g., Johnson and Leslie (1982), Milardo et al. (1983), and Roberts et al. (2009). Capacity constraints have also recently received special attention by more theoretical approaches in social and economic networks, e.g., Bloch and Dutta (2009) and König et al. (2010).

networks in the presence of capacity constraints. In consonance with the documented diverse assortative motivations, we allow for (exogenous) assortative interests to vary generally. We then study how such interests lead to certain (endogenous) homophilic patterns. We are particularly interested in understanding if and why some heterophilic features could arise even when individuals have strong assortative preferences. Finally, we investigate the efficiency properties of networks when people are constrained in making friends. Our main contribution is to complement the growing literature on homophilic features of friendship networks. Allowing assortative interests to vary generally in a context in which individuals are naturally capacity constrained enables us to undertake a comprehensive exploration of stability and efficiency properties.

Agents are distinguished according to a certain (extrinsic) characteristic. Then, they invest quantities of an available resource to form friendship *links* and to build the *qualities* of such links. The linkage technology is monotone and, crucially, it features strategic independence between the investments made by the agents. Importantly, considering instead strategic complementarities in the linkage technology would naturally lead to that agents benefit from having dissimilar friends. This would directly make it easer for heterophilic patterns to arise even in the presence of strong assortative preferences. Since we want to understand how heterophilic features may coexist with strong assortative interests, we opt for isolating the role of preferences from other counter-effects that may be derived from the linkage technology. Thus, the assumption of strategic independence across investments made in the linkage technology stands as a crucial one to address the questions explored in this paper. The preferences of the agents depend only on the aggregate qualities of their links in the friendship network. We consider a (common) level of assortative interests in the agents' preferences which is captured by a certain (exogenous) parameter.

Stability of a friendship network requires that no agent has (strict) incentives to change her aggregate investments in similar and dissimilar people (*robustness against individual deviations*) and, furthermore, that no pair of agents benefit (strictly) by changing their friendship investments (*robustness against bilateral deviations*). Our stability notion reduces critically the set of equilibria relative to the analysis where only individual deviations (i.e., Nash stability) would be considered. In this way, we can conduct a meaningful analysis of stability, in which comparative-statics exercises can be performed.

We obtain a positive relation between (exogenous) assortative interests and (endogenous) homophilic stable patterns. In addition to this natural implication, the presence of capacity constraints leads to other interesting insights in which homophily patterns are coupled with heterophilic features even when assortative interests are strong. First, although the stability notion that we use refines most concepts commonly proposed in the literature,[4] *multiple networks, with very different homophilic features, may arise as stable for assortative interests that are not too extreme*. Secondly, we put forward a crucial insight about the incentives of agents to sustain friendship links, which we term as "pre-

[4] In particular, it refines the notions of Nash stability, Pairwise stability, and Pairwise Nash stability.

mium of mutual efforts." This premium of mutual efforts tells us a lot about the friendship connections in stable patterns that feature extreme forms of homophily or heterophily. In particular, for a network to feature high homophily, the existing links between agents with different characteristics must be relatively intense as well. In short, *heterophilic connections of relatively high quality must be present in extreme forms of homophilic patterns*.[5] Another interesting implication of the premium of mutual efforts is that, in stable patterns, *individuals coordinate in a way such that only one friend in each relationship acts as "main sponsor" while the other "free rides."*

The logic behind the premium of mutual efforts is as follows. Under a simple monotone additive-linear linkage technology, both agents in any given pair can benefit strictly if they redirect simultaneously into each other efforts devoted to other friends (outside from the pair). As a consequence, if the effort that any agent could make in some other agent were unbounded, then no pattern would be stable.[6] However, since the amount of investments that can be made (and received) for each particular relation are naturally bounded, the premium of mutual efforts ceases to have effect if one friend is already saturating what she can invest in the other. This leads to the insight that a stable friendship networks requires that, in each pair of different agents, at least one of them invests with full intensity into the other. The capacity constraint that the agents face in forming links plays a key role in this insight.

Since preferences and capacity constraints are homogenous across agents, the convexity of such preferences lead to the key implication that an efficient friendship network must feature a particular form of uniform aggregate qualities within the agents that have a common characteristic. This implication highlights a source for inefficiency of stable networks which is based on two requirements. First, stability requires that each agents obeys individually her incentives according to the given level of assortative interests (under the restriction of their capacity constraints). Secondly, proposing particular stable patterns requires the construction of *minimal sets of full-intensity investments between pairs of agents* in order to avoid the effects of the premium of mutual efforts. We are able to construct minimal sets of full-intensity investments between agents of a common characteristic that guarantee efficiency of some extreme forms of heterophilic patterns. Similarly, we propose full-intensity investments between agents of different characteristics that ensures efficiency of some extreme forms of homophilic patterns, under the condition that the sizes of the two groups coincide. Finally, for intermediate levels of assortative interests, we identify stable patterns that feature intermediate levels of homophily and, for an intuitive class of such networks, we propose full-intensity investments that guarantee the efficiency of such patterns.

Although any (non-extreme) level of assortative interests may give rise to multiple

[5] The analog message follows for populations that feature extreme forms of heterophilic patterns.

[6] Intuitively, Anne and Bob could always diminish their efforts with respect to some other friends and gain by using the so saved resources in improving jointly their own relationship. But then Bob and Charles could do the same, and so on, endlessly.

classes of stable friendship patterns in our setup, we are able to identify particular bounds under which unique classes of homophilic patterns always exists as stable. Multiplicity of stable networks is a common feature in the literature on social networks. Therefore, the bounds that we provide—which ensure both existence and uniqueness—can have useful implications for statistical work. Given that extreme forms of homophilic behavior are extensively documented by the empirical literature, and the pertinent data can be easily obtained, an econometrician can use our results on the uniqueness of patterns to infer properties about underlying assortative interests in many practical scenarios.

The article is organized as follows. Section 2 outlines the baseline model. Section 3 analyzes the properties of stability of friendship networks and Section 4 focuses on their efficiency features. Section 5 comments on literature connections and Section 6 concludes. Appendix A and Appendix B provide formal expressions and proofs omitted in the main text.

2 A Model of Friendship Relationships

There is a *population* $N \equiv \{1, \ldots, n\}$ of agents that can be distinguished according to a certain (extrinsic) characteristic—e.g., ethnicity, education, profession, or age. Each agent has a *type* $\theta \in \Theta \equiv \{A, B\}$ that captures the characteristic. Based on θ, the entire population N consists of two *groups of people* N_θ with respective sizes $n_\theta \equiv |N_\theta|$, for $\theta \in \Theta$, so that $N = N_A \cup N_B$ and $n = n_A + n_B$.[7]

We assume that $n_\theta \geq 3$ for each $\theta \in \Theta$ and, without loss of generality, set $n_A \geq n_B$ throughout. When considering a given a type $\theta \in \Theta$, we will typically use θ' to refer to the alternative type $\theta' \neq \theta$. Also, for agent i of type θ, we will use the short-hand notation $N_\theta^i \equiv N_\theta \setminus \{i\}$ to indicate the group of agents, other than herself, that have her own characteristic.

2.1 Friendship Networks

People make investment decisions to build up links that, in turn, give rise to friendship networks. A *friendship network* g is a collection $g \equiv \{g_{ij} \in [0,1] \mid i, j \in N\}$ of *linkage qualities* $g_{ij} \in [0,1]$ for each pair of agents $i, j \in N$. A linkage quality g_{ij} captures the quality of the link that goes from agent i to agent j under network g. We consider *undirected networks* in which links are bidirectional so that, by construction, $g_{ij} = g_{ji}$ for each pair of agents $i, j \in N$. We consider that each agent is linked to herself with full quality, i.e., $g_{ii} = 1$. We use G to denote the set of all possible friendship networks.

[7] Although the model considers two population groups, the main qualitative implications continue to follow under an arbitrary number of groups.

2.2 Linking Decisions

Individuals make their linking decisions in a (simultaneous-move) network formation game. Each agent i makes simultaneously an *investment effort* $x_{ij} \in [0,1]$ to build up a friendship link with each other agent $j \neq i$ in the population.[8] An *investment strategy* for an agent i is a vector $x_i \equiv (x_{ij})_{j \neq i} \in [0,1]^{n-1}$. Let $x \equiv (x_i)_{i \in N} \in [0,1]^{n(n-1)}$ be a *strategy profile*. As usual, x_{-i} will denote a combination of strategies for all individuals other than agent i. Similarly, let x_{-i-j} denote a combination of strategies for all individuals excluding the pair of (different) individuals i and j. Thus, we can express a strategy profile x either as $x = (x_i, x_{-i})$ or as $x = (x_i, x_j, x_{-i-j})$, for $j \neq i$.

Investments in a friendship connection determine the quality of the link according to a simple additive-linear technology.

ASSUMPTION 1. Given a strategy profile x, the *linkage quality* $g_{ij}(x) = g_{ji}(x)$ of the connection between agents i and j is given by

$$g_{ij}(x) \equiv (1/2)\left[x_{ij} + x_{ji}\right]. \tag{1}$$

In particular, the formation of a friendship relation does not require a positive effort by both agents, though its quality is enhanced when both contribute. An analogous additive-separability assumption is made in Bloch and Dutta (2009).

Let $g(x)$ denote the friendship network induced by the profile x according to the technology described by Eq. (1) above. Given a strategy profile x and an agent $i \in N$, let $N_i(x) \equiv \{j \in N \setminus \{i\} \mid x_{ij} = 1\}$ be the set of agents that receive full-intensity investments from agent i under the profile x. Also, for agent i of type θ, let the quantity $s_i(x) \equiv \sum_{j \in N_\theta^i} g_{ij}(x)$ be the aggregate quality of the links that connect agent i to all other same-type agents, and, analogously, let $d_i(x) \equiv \sum_{j \in N_{\theta'}} g_{ij}(x)$ describe the total quality of the links that connect agent i to all different-type agents. When no reference need be made to the underlying strategy profile x, we will drop the x argument and simply write s_i and d_i. Let then $S_i \equiv [0, n_\theta - 1]$ and $D_i \equiv [0, n_{\theta'}]$ be the sets of possible total qualities, respectively, of same-type and different-type links for agent i of type θ. Allowing for $x_{ij} \in [0,1]$ leads to that the variables $s_i \in S_i$ and $d_i \in D_i$ are non-negative real numbers.

2.3 Preferences

The preferences of an individual i over networks are described by a function $\pi_i : G \to \mathbb{R}_+$. We assume that each agent i cares only about the total qualities (s_i, d_i) associated to her friendship links.[9] Specifically, we consider that the function π_i has the form $\pi_i(g(x)) =$

[8] The assumption that $x_{ij} \in [0,1]$ allows for "infinitesimal" investment efforts.

[9] We are thus not considering other plausible ways in which people could in principle care about the architecture of the resulting friendship network $g(x)$. In particular, agents do not care about the identity of

$u(s_i(x), d_i(x))$, where $u : S_i \times D_i \to \mathbb{R}$ captures the utility $u(s_i, d_i)$ that any agent i receives from the aggregate qualities (s_i, d_i) of her same-type and different-type friendship links. The function u is common across agents. Assuming that agents do not care about the entire architecture of the network is relatively common in the literature on friendship connections—e.g., Currarini et al. (2009); Boucher (2015); Currarini et al. (2017), among others. Under the above considerations, we assume

ASSUMPTION 2. For each agent $i \in N$, the utility function u is smooth and satisfies:

(1) $u(0,0) = 0$ and $u(s_i, d_i) \geq 0$ for each $(s_i, d_i) \neq (0,0)$.

(2) $u(s_i, d_i)$ is strictly increasing in (s_i, d_i).

(3) $u(s_i, d_i)$ is strictly concave in (s_i, d_i).

(4) There is a given cutoff proportion $\beta \in (0, +\infty)$ of qualities of different-type (relative to same-type) friendship links such that

  (a) $\partial u(s_i, d_i)/\partial s_i = \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i = \beta$;
  (b) $\partial u(s_i, d_i)/\partial s_i > \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i > \beta$;
  (c) $\partial u(s_i, d_i)/\partial s_i < \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i < \beta$.

Assumption 2−(1) is just for normalization. Assumption 2−(2) imposes monotonicity on the utility that each agent receives from the qualities of her friendship links. Geometrically, in the (s_i, d_i) space, the utility from any investments in friendship links increases in any ray that departs from the origin. Assumption 2−(3) imposes convexity on each agent's preferences over the (s_i, d_i) space of total friendship qualities.

Assumption 2−(4) is key to describe the way in which agents could either be (relatively) more interested in mating either same-type or different-type individuals. In short, the condition describes whether agents have either assortative or disassortative interests, as well as the degree of such interests. In particular, Assumption 2−(4) establishes that (a) there is a fixed fraction $\beta = d_i/s_i$—which geometrically corresponds to the slope of a ray going out of the origin in the space (s_i, d_i)—such that the marginal utilities from linking with either type of agents are equal. Given this cutoff value β, then (b) if the proportion of qualities of different-type links (relative to same-type links) lies above the required fraction β, then the marginal utility from additional qualities of different-type links becomes lower than the marginal utility from same-type links. The converse condition is described by condition (c).[10]

the agents they are linked to, neither about the features of their indirect connections (i.e., features along the paths of friends of own friends).

[10] This assumption can be equivalently interpreted in terms of the marginal rate of substitution of the utility function u between the aggregate qualities s_i and d_i. Geometrically, the conditions put structure on the slopes of the agent's indifference curves in the (s_i, d_i) space.

Intuitively, parameter β captures the (common) level of assortative interests in the population. Values of β in the interval $(0,1)$ correspond to situations where people lean relatively more towards assortativity, whereas values of β in the interval $(1,+\infty)$ describe situations where disassortative interests prevail.[11] As mentioned in the Introduction, we can interpret assortative interests as being based on socialization motivations and disassortative interests as relying more on the sort of collaboration motivations.

2.4 Capacity Constraints

An important consideration of the model is the presence of *capacity constraints* over the total investments in friendship qualities. Each agent has a total resource (e.g., time, information-load limits) $R > 0$ (captured by a positive integer) to invest in friendship links with others. We assume

ASSUMPTION 3. Each individual $i \in N$ is constrained over her friendship investments x_{ij} according to the restriction $\sum_{j \neq i} x_{ij} \leq R$, for a certain bound $R \in \{n_A + 1, \ldots, n-1\}$.

Since $R \geq n_A + 1 > n_A$, each agent is able to invest with full intensity in links to all other agents from either group, N_A or N_B, separately. Also, since $R < n-1$, no agent is able to invest with full intensity in links to all the remaining agents in the population.[12] We consider that the possible values of the total resource R are integer numbers for technical (and expositive) reasons.[13]

Let $X_i \equiv \{x_i \in [0,1]^{n-1} \mid \sum_{j \neq i} x_{ij} \leq R\} \subset [0,1]^{n-1}$ be the set of agent i's *investment strategies under capacity constraints* and let $X \equiv \times_{i \in N} X_i \subset [0,1]^{n(n-1)}$ be the *set of all possible investment profiles under capacity constraints*.

2.5 Stability Notion

Let $\Gamma \equiv < N, \Theta, X, (\pi_i)_{i=1}^n >$ denote the network formation game that we have described. To explore stable friendship networks, we consider the *weak bilateral equilibrium (wBE)* stability concept proposed by Boucher (2015).

DEFINITION 1. A *weak bilateral equilibrium (wBE)* of the network formation game Γ is a strategy profile x^* that satisfies:

[11] While lower values of the cutoff ratio $\beta \in (0,1)$ describe higher levels of assortative interests, higher values of $\beta \in (1,+\infty)$ describe higher levels of disassortative interests. An example of a preference specification u that satisfies all the conditions required by Assumption 2 is that given by a Cobb-Douglas function $u(s_i,d_i) = s_i^a d_i^b$ such that $a > 0$, $b > 0$, and $a+b < 1$. In this case, the level of assortative interests β described in Assumption 2$-$(4) is equal to $\beta = b/a$.

[12] This is an obvious requirement to keep the model interesting under strictly monotone preferences.

[13] Such a discrete set of possible values for R allows us to have a clear description of how investment in friendship links can be allocated in the presence of monotone preferences and capacity constraints.

1. *robustness against unilateral deviations:* for each individual $i \in N$, we have
   $\pi_i(g(x^*)) \geq \pi_i(g(x_i, x^*_{-i}))$ for each $x_i \in X_i$;
2. *robustness against bilateral deviations:* for each pair of (different) individuals $i, j \in N$, we have
   $\pi_i(g(x_i, x_j, x^*_{-i-j})) > \pi_i(g(x^*)) \;\Rightarrow\; \pi_j(g(x_i, x_j, x^*_{-i-j})) \leq \pi_j(g(x^*))$ for each $x_i \in X_i$ and $x_j \in X_j$.

A network g is a *stable friendship network* if there is a weak bilateral equilibrium x^* of the network formation game Γ such that $g = g(x^*)$.

Condition 1. of Definition 1 is the best-response requirement of the Nash stability notion. Condition 2. adds then the requirement that a wBE be immune also against any possible bilateral deviation that be *strictly* beneficial to *both* agents in the pair.

The notion of wBE weakens the concept of bilateral equilibrium due to Goyal and Vega-Redondo (2007). Yet, it refines most stability notions commonly used in the literature on network formation. In particular, wBE refines Nash stability (proposed by Myerson (1991)), Pairwise stability (proposed by Jackson and Wolinsky (1996)), and Pairwise Nash stability (which combines both the Nash and the Pairwise stability requirements).[14]

3 Stability of Friendship Networks

We follow a two-step strategy to explore stability features of friendship networks. In the first step, we characterize (in Lemma 1) the optimal investment strategy of any given agent as a best-response to the investments strategies chosen by the rest of individuals. By considering networks where all agents best-reply to the rest of agents, we derive Nash stable networks, as required by condition 1. of our stability notion (Definition 1). In the second step, we identify (in Lemma 2) —and thereby rule out—plausible profitable bilateral deviations from any given network that is already robust against unilateral deviations. Accordingly, these two steps provide conditions to identify stable networks according to both 1. and 2. of Definition 1.

3.1 Step I–Unilateral Optimal Decisions

Fig. 1 illustrates geometrically the decision problem that each agent faces when she cares only about her unilateral incentives. Given our assumptions on preferences, it is useful to work with a given agent i's (unilateral) problem directly in terms of the variables s_i and

[14] Interestingly, any further refinement of the notion of wBE would lead to that no stable network exists in our setup. In this sense, our approach to stability seeks to meaningfully reduce the multiplicity of stable networks down to a minimum

d_i.[15] For agent i of type θ, let $I_i^s(x_{-i}) \equiv (1/2)\sum_{j \in N_\theta^i} x_{ji}$ and $I_i^d(x_{-i}) \equiv (1/2)\sum_{j \in N_{\theta'}} x_{ji}$ be the (normalized) *total incoming intensities* that agent i receives, respectively, from same-type and different-type people under the combination x_{-i}.[16] Then, for a fixed x_{-i}, each agent i wishes to choose x_i so as to maximize $u(s_i, d_i)$ in a way such that the induced total qualities $s_i = s_i(x_i, x_{-i})$ and $d_i = d_i(x_i, x_{-i})$ satisfy the restrictions (displayed in Fig. 1):

(i) the green dotted line gives us the constraint that the sum of induced total qualities s_i and d_i does not exceed the available resource R plus the investments made by the rest of agents, I_i^s and I_i^s;

(ii) the horizontal black line gives us the restriction that s_i must lie in the interval $[I_i^s, (n_\theta - 1)/2 + I_i^s]$. Here, the lower bound corresponds to the situation in which agent i does not invest in forming same-type links and the upper bound corresponds to the situation in which agent i invests as much as she can to form same-type links;

(iii) analogously to restriction (ii), the vertical black line gives us the restriction that d_i must lie in the interval $[I_i^d, n_{\theta'}/2 + I_i^d]$.

The rays in red correspond to three possible levels of assortative interests $\beta > \beta' > \beta''$, as described by Assumption 2-(4). In the figure, β gives us disassortative interests, whereas β' and β'' give us two different levels of assortative interests.

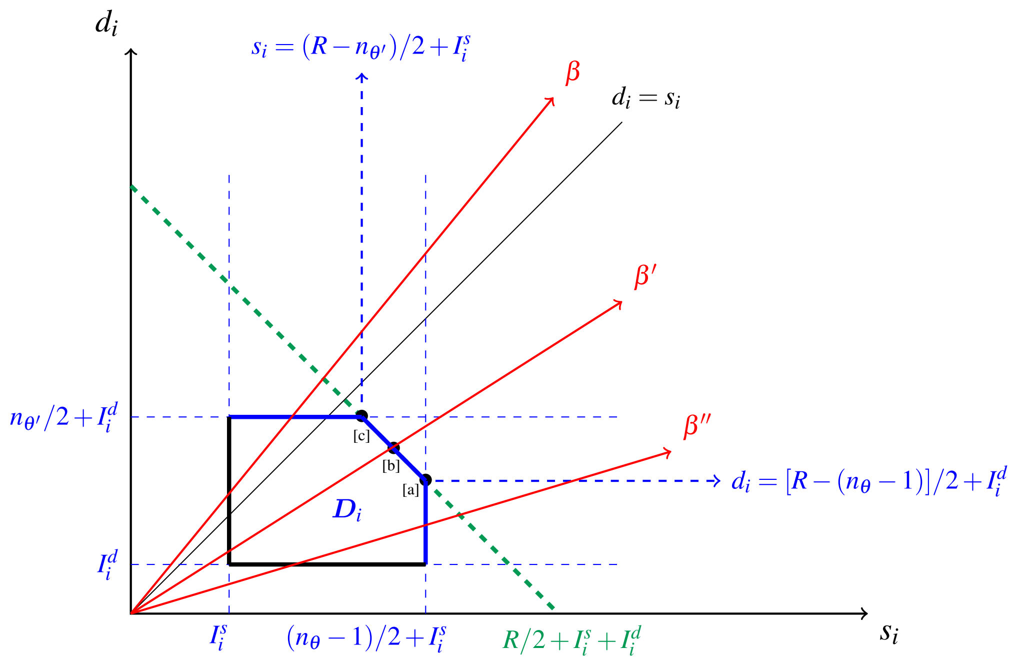

Figure 1 – Optimal choices of (s_i, d_i) for a given x_{-i}^*.

[15] The formal expression of the individual maximization problem is derived in Eq. (9) in Appendix A.

[16] We will drop the x_{-i} argument when no reference need be made to the underlying strategy combination. Note that $s_i = (1/2)\sum_{j \in N_\theta^i} x_{ij} + I_i^s$ and $d_i = (1/2)\sum_{j \in N_{\theta'}} x_{ij} + I_i^d$ for each strategy profile $x \in X$.

Analyzing agent i's (unilateral) problem boils down to studying how the linear constraints that describe the feasible set D_i of the problem bind in plausible solutions. The monotonicity condition of Assumption 2−(2), ensures that agent i wishes to choose x_i so as to induce the highest possible qualities s_i and d_i. Thus, the solution lies on the green dotted line, where agent i exhausts her resource R. Hence, we have that

$$s_i + d_i = R/2 + I_i^s + I_i^d. \tag{2}$$

For a fixed $x^*_{-i} \in X_i$, the optimal choice (s_i^*, d_i^*) of agent i is described by the point where the highest indifference curve along the corresponding ray β intersects the feasible set D_i. Since u is smooth and concave, the solutions to this problem can always be obtained as captured by [a], [b], and [c] in Fig. 1. Such solutions describe the three key qualitative cases that can take agent i's (unilaterally optimal) investment. Notice that [a] and [c] are corner solutions where agent i invests with full intensity in same-type and different-type individuals, respectively, whereas [b] is an interior solution.

The optimal choice of agent i is summarized Lemma 1 below. It is useful to set for agent i of type θ and for a fixed x_{-i}, the following values for the cutoff level β:

$$\underline{\beta}(\theta; x_{-i}) \equiv \frac{R - (n_\theta - 1) + 2I_i^d(x_{-i})}{(n_\theta - 1) + 2I_i^s(x_{-i})} \text{ and } \overline{\beta}(\theta; x_{-i}) \equiv \frac{n_{\theta'} + 2I_i^d(x_{-i})}{R - n_{\theta'} + 2I_i^s(x_{-i})}. \tag{3}$$

LEMMA 1. Assume Assumption 1—Assumption 3, and consider a preference specification u. Take a given agent $i \in N$, and take a given strategy combination x^*_{-i} chosen by the agents other than i. Consider the unilateral problem of agent i specified in Eq. (9). Then, the solutions to such a linear problem are described by:

$$s_i^* = \frac{(n_\theta - 1)}{2} + I_i^s,\ d_i^* = \frac{R - (n_\theta - 1)}{2} + I_i^d \text{ if } 0 < \beta \leq \underline{\beta}(\theta; x^*_{-i}). \tag{a}$$

$$s_i^* = \left(\frac{1}{1+\beta}\right)\left(\frac{R}{2} + I_i^s + I_i^d\right),\ d_i^* = \beta s_i^* \text{ if } \underline{\beta}(\theta; x^*_{-i}) \leq \beta \leq \overline{\beta}(\theta; x^*_{-i}); \tag{b}$$

and

$$s_i^* = \frac{R - n_{\theta'}}{2} + I_i^s,\ d_i^* = \frac{n_{\theta'}}{2} + I_i^d \text{ if } \beta \geq \overline{\beta}(\theta; x^*_{-i}); \tag{c}$$

If each agent $i \in N$ chooses her investment strategy $x_i^* \in X_i$ as described in Lemma 1, for each given $x^*_{-i} \in X_{-i}$, then the resulting network $g = g((x_i^*, x^*_{-i}))$ is Nash stable. Let us denote by $NS(u) \subset G$ the set of Nash stable networks for a preference specification given by u. Existence of Nash stable networks in the proposed network formation game Γ, for any given u, follows directly from the following modeling choices: (1) $x_i \in [0,1]$ for each agent i, (2) the presence of capacity constraints, and (3) the assumption that u is smooth and concave.[17]

[17] Formal arguments are provided in Appendix A.

3.2 *Step II–Bilateral Optimal Decisions*

Upon ruling out unilateral deviations, stable networks follow by preventing bilateral deviations as well, as required by condition 2. of Definition 1. Lemma 2 provides the key necessary and sufficient condition for a Nash stable network $g(x) \in NS(u)$ to be immune against bilateral deviations.

LEMMA 2. Assume Assumption 1—Assumption 3, and consider a preference specification u. Consider a strategy profile $x \in X$ that induces a Nash stable friendship network $g = g(x) \in NS(u)$. Then, any given pair of two (different) agents $i, j \in N$ does not have incentives to bilaterally deviate from the profile x, as described by condition 2. of Definition 1, that is,

$$\pi_i(g(x'_i, x'_j, x_{-i-j})) > \pi_i(g(x)) \;\Rightarrow\; \pi_j(g(x'_i, x'_j, x_{-i-j})) \leq \pi_j(g(x))$$

for each $x'_i \in X_i$ and $x'_j \in X_j$ if and only if $j \notin N_i(x) \Rightarrow i \in N_j(x)$.

The logic behind the condition derived in Lemma 2 is as follows. Under the maintained assumptions, if we start from a strategy profile that induces a Nash stable network, then there exists a unique class of bilateral deviations that could be strictly beneficial to each agent from any given pair of agents in the society. This class of (potentially) profitable deviations is based on each of the two agents from a fixed pair being able to decrease their investments in some other agents by some arbitrary amounts. Then, this class of deviations requires that they could invest the so saved amounts into each other. Because of the simple additive-linear technology for generating linkage quality considered in Eq. (1), this class of deviations would clearly be strictly profitable for both agents.

At a more intuitive level, the suggested (potential) deviations capture natural situations where two friends benefit from "synergies" by mutually increasing the efforts they devote to their own relationship. We term the incentives behind this class of (potential) deviations as "premium of mutual efforts" because both agents in a given pair benefit strictly by mutually redirecting third-party investments into each other.

Furthermore, under our monotonicity assumptions in the presence of capacity constraints, the only class of bilateral deviations—starting from a Nash stable network—that could result strictly beneficial to both agents in a given pair is the one described above. More specifically, the only way of in which the two members of a given pair can benefit strictly is by redirecting third-party investments. All that any other type of bilateral deviations could attain is to maintain indifferent (at least) one of the agents in the pair.

To ensure then that a friendship Nash stable network is immune against this class of deviations, we need to restrict attention to strategy profiles in which at least one agent in each possible pair cannot increase any further her investment in the other agent, as stated in Lemma 2. As a consequence, in order to propose a stable friendship pattern, we need to construct a minimal set of full-intensity investments across all agents. An intuitive insight that stems form such minimal sets, in the presence of (common) capacity constraints, is then that (in stable networks x) each particular friendship relationship $g_{ij}(x)$ is mainly sponsored by only one of the two friends, say i, while the recipient j of the full-intensity investment sponsors in turn other relationships. As the capacity constraints tighten, the recipient j of the full-intensity effort in each particular relationship reciprocates less with her friend i. Only by doing so, the recipient j is able to save amounts of the resource R to meet the required full-intensity investments in other agents $k \neq i$.

The description that we give of the premium of mutual efforts, and the role that it plays in stability, is also robust to alternative linkage technology specifications, so long as such alternative technologies be linear with uniform slopes across the investments of all agents in the population. Under such technologies, for situations where no agent from a given pair invests fully in the other agent, both agents would benefit strictly by redirecting third-party investments. More specifically, consider a technology given by

$$g_{ij}(x) = A + Bx_{ij} + Bx_{ji},$$

where $A \geq 0$, $B > 0$. Suppose, without loss of generality, that agents i and j have the same type. Consider now situations where $x_{ij} < 1$ and $x_{ji} < 1$. Then, suppose that agent i decreases her investment in some other agent $k \notin \{i, j\}$ of her same type, by an amount $\varepsilon_i > 0$ and that j decreases her investment in some other agent $l \notin \{i, j\}$ of her same type, by an amount $\varepsilon_j > 0$, where it may well be that $k = l$. This is enabled by the assumption that $n_\theta \geq 3$ for each type $\theta \in \Theta$. Consider that the two agents i and j invest the saved amounts ε_i and ε_j into each other. Notice then that for agent i, s_i increases by a net amount of $B\varepsilon_j$ and for agent j, s_j increases by a net amount of $B\varepsilon_i$. Also, d_i and d_j remain unchanged. Thus this deviation is strictly profitable to both agents i and j. Similar arguments can be provided for the case where i and j have different types.

Our description of the premium of mutual efforts, however, does not go through under more general specifications of linkage quality technology. For instance, the identified class of (potentially) beneficial bilateral deviations does not work as considered in this paper if either $g_{ij}(x)$ were linear in x_{ij} and x_{ji} with different slopes, or if $g_{ij}(x)$ were strictly concave or convex in the agents' investments. Nevertheless, our particular assumption of linear technology (with the form given by Eq. (1)) gives us a reasonable and simple formulation of how investments produce linkage quality for a continuous investment choice. Given the degree of generality that we consider on the utility function u, more complex technology specifications would render intractable the analysis of general properties of stable friendship patterns[18]

Finally, let us comment on the existence of stable networks in the proposed model.[19] In some parts of the analysis, we deal with the fact that the sizes of the population groups

[18] We discuss this issue further in Section 6.

[19] As argued earlier (in Subsection 3.1), existence of Nash stable patterns is in fact ensured in our setting for each possible tuple (β, R, n_A, n_B).

N_A and N_B can be either odd or even. It is then useful to specify the number

$$\alpha(r) \equiv \begin{cases} r/2 \text{ if } r \text{ is even;} \\ (r-1)/2 \text{ if } r \text{ is odd} \end{cases}$$

for any given integer $r > 1$. The number $\alpha(r)$ accounts for either half of r or half of $r-1$, depending on whether r is an even or an odd integer, respectively. A necessary condition for the investment requirements in Lemma 2 to be satisfied is that the size of the resource available to the agents be sufficiently large, in particular, it must be that $R \geq \alpha(n)$. Then each agent can take on the burden of full-intensity investments for (approximately) half of the population. Notice that condition $R \geq \alpha(n)$ is guaranteed by Assumption 3 since $n_A + 1 > \alpha(n)$. This, however, does not directly ensure the existence of a stable friendship network for each possible tuple (β, R, n_A, n_B). The presence of (homogeneous) capacity constraints, combined with discrepancies between the groups sizes, may conflict crucially with the incentives described by the level of assortative interests β.

We guarantee existence (and uniqueness in some cases) of stable friendship networks, for each possible tuple (β, R, n_A, n_B), for relatively high or low (dis)assortative interests β. We make no claims regarding existence of stable patterns for "intermediate" levels of the assortative interest β. To illustrate the difficulties that may arise regarding existence for intermediate levels of assortative interests, consider a population in which the sizes of the two groups are very different and suppose that the capacity constraints are relatively tight. Suppose that the level of assortative interests is intermediate and, accordingly, consider a resulting Nash stable network $g(x) \in NS(u)$ such that some agents have unilateral incentives to invest mainly in same-type fellows (as in [a] of Fig. 1) while other agents have unilateral incentives to invest both in same-type and different-type agents (as in [b] of Fig. 1). When the capacity constraints are as tight as possible (according to Assumption 3), each agent is able to invest with full intensity in only $\alpha(n)$ other agents. We can intuitively observe then that some agents may not be able to simultaneously comply with their individual assortative interests and, at the same time, meet their shares of full-intensity contributions, which are required to prevent bilateral deviations. For instance, if agents of the smaller group want to behave unilaterally as in as in [b] of Fig. 1, then they wish to invest large amounts in relations with members of a much larger group. Then, they may end up not having enough resource so as to comply with the overall minimal full-intensity requirements. As a consequence, it might well be the case a network $g(x) \in NS(u)$ be not immune against beneficial bilateral deviations.

Let us use $S(u) \subset NS(u)$ to denote the set of stable friendship networks for a given preference specification u. We will be more specific as to when we can guarantee existence of stable patterns in Subsection 3.5.

3.3 *Classifying Networks in Terms of their Friendship Patterns*

On one extreme, we consider networks in which all agents invest with full intensity in links to all others of their same type. Given this, each agent devotes the remaining of her resource R to different-type agents. We refer to such networks as *maximally homophilic* networks. On the other extreme, we consider networks in which all agents invest with full intensity in different-type agents. Then, the agents devote the remaining of their resources to links to agents of their same type. We refer to such networks as *minimally homophilic* networks.

DEFINITION 2. Consider a strategy profile x that induces a friendship network $g = g(x)$. Then,

(a) the network g is said to be *maximally homophilic* if for each agent $i \in N$ of type θ, and for each type $\theta \in \Theta$, we have $N_\theta^i \subseteq N_i(x)$, and

(b) the network g is said to be *minimally homophilic* if for each agent $i \in N$ of type θ, for each type $\theta \in \Theta$ and for the alternative type $\theta' \neq \theta$, we have $N_{\theta'} \subseteq N_i(x)$.

We take the simple approach to regard a friendship network as *partially homophilic* whenever it is neither maximally nor minimally homophilic.

DEFINITION 3. Consider a strategy profile x that induces a friendship network $g = g(x)$. The network g is said to be *partially homophilic* if for some type $\theta \in \Theta$, we have that

(a) there is some agent i of type θ such that $N_i(x) \subset N_\theta^i$, with $N_i(x) \neq N_\theta^i$, and

(b) there is some agent j of type θ such that $N_j(x) \subset N_{\theta'}$, with $N_j(x) \neq N_{\theta'}$.[20]

3.4 *Stable Friendship Networks*

The previous insights about unilateral and bilateral optimal choices allow us to explore stable friendship networks. It will be useful to consider the following relevant values of the cutoff level β of assortative interests, which depend on the primitives of the model.

$$\beta_L \equiv \frac{R-(n_A-1)}{2(n_A-1)}, \; \beta_l \equiv \frac{nR - n_A(n_A-1) - n_B(n_B-1)}{2n_A(n_A-1)}, \quad \beta_h \equiv \frac{n_A}{R-n_A}, \text{ and } \beta_H \equiv 2\beta_h. \tag{4}$$

The conditions provided by Proposition 1 below characterize strategy profiles that induce stable maximally homophilic networks. Given a strategy profile $x \in X$, the following upper bound

$$\hat{\beta}(x) \equiv \inf_{i \in N_\theta, \theta \in \Theta} \frac{R-(n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ji}}{2(n_\theta - 1)}$$

[20] Note that the agents i of (a) and j of (b) in Definition 3 are not required to be different agents.

on the level of assortative interests of the population will be useful to understand how assortative interests lead to maximally homophilic networks.

PROPOSITION 1. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x a strategy profile that induces a maximally homophilic friendship network $g = g(x)$. Then, the network g is stable, i.e., $g \in S(u)$, if and only if:

1. *Robustness against unilateral deviations:* the level of assortative interests of the population described by β is sufficiently high, with the particular form given by $\beta \leq \hat{\beta}(x)$.

2. *Robustness against bilateral deviations:* provided that the resource R is sufficiently large, with the particular form given by $R \geq (n-1) - n_A n_B/n$, then for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$.

Proposition 1 characterizes investment profiles that lead to maximally homophilic networks, in terms of the primitives n_A, n_B, β, R. Interestingly, provided that the resource R is relatively large, even in the presence of high assortative interests, patterns of extreme homophily features can be sustained as stable ones only if links of a certain (relatively high) quality between agents of different types arise as well. In particular, there must exist a link between each pair (i, j) of different-type agents with quality g_{ij} no less than $1/2$. In short, a certain degree of qualities of heterophilic relations is necessary to sustain maximally homophilic stable networks.

In a natural manner, maximally homophilic networks require that the level of assortative interests of the agents be sufficiently high. Nonetheless, multiple networks, not all of them necessarily being maximally homophilic, may arise as stable ones for the levels of assortative interests captured by Proposition 1. Corollary 1 gives us a bound on the level of assortative interests that guarantees the existence of only maximally homophilic networks.

COROLLARY 1. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x be a strategy profile that induces a friendship network $g = g(x)$. Provided that $R \geq (n-1) - n_A n_B/n$, suppose that x satisfies that for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$. Then, if the level of assortative interests in the population is sufficiently high, with the particular form given by $\beta < \beta_L$, where β_L is the bound on β specified in Eq. (4), then the unique class of stable networks consists entirely of maximally homophilic netwoks.

The sufficient condition stated in Corollary 1 follows directly by considering a bound β_L on the cutoff values $\hat{\beta}(x)$ such that, if $\beta < \beta_L$, then each agent would choose to invest with full intensity in all same-type others as captured by [a] of Fig. 1. The value of β_L is derived by noticing that the bound $[R - (n_\theta - 1)]/2(n_\theta - 1)$, would correspond to a (hypothetical) situation where an agent $i \in N_\theta$ would receive no investments whatsoever from different-type agents and full investments from all other same-type agents, that is,

$\sum_{j\in N_{\theta'}} x_{ji} = 0$ and $\sum_{j\in N_\theta^i} x_{ji} = (n_\theta - 1)$. Then, if $\beta < [R-(n_\theta-1)]/2(n_\theta-1)$, even such an agent would optimally behave as captured by [a] in Fig. 1.

We now use the insights from Proposition 1 to explore further certain features of maximally homophilic stable networks. Motivated by the consideration that all agents in the society face a common available resource R, we pay special attention to stable networks in which the burden of full-intensity investments between the agents that belong to different groups is distributed across the agents in a relatively uniform way. Specifically, Corollary 2 gives us conditions under which a class of maximally homophilic networks, with certain symmetries in the amounts invested by the agents, arise as stable ones. To this end, it is convenient to introduce first the details of a certain partition of any of the two population groups N_θ, for $\theta \in \Theta$.

OBSERVATION 1. Upon relabelling the names of the agents (say, switching indexes from i to i_k and j_k), let us partition each of two groups N_θ, for $\theta \in \Theta$, into two sets, N_θ^L and N_θ^H, according to: (a) N_A is partitioned into $N_A^L = \{i_1,\dots,i_{\alpha(n_A)}\}$ and $N_A^H = \{i_{\alpha(n_A)}+1,\dots,i_{n_A}\}$, whereas (b) N_B is partitioned into $N_B^L = \{j_1,\dots,j_{\alpha(n_B)}\}$ and $N_B^H = \{j_{\alpha(n_B)}+1,\dots,j_{n_B}\}$.

The partition specified in Observation 1 separates each group N_θ into two subgroups, N_θ^L and N_θ^H, of the same size if the number of agents n_θ in the group is even. If the number of agents in the group N_θ is odd, then the set N_θ^H contains just one more agent than the set N_θ^L. Thus, the sizes of N_θ^L and N_θ^H are as similar as possible. Provided that the level of assortative interests and the size of the resource are sufficiently large, Corollary 2 describes a class of maximally homophilic networks in which each agent from each group N_θ takes on the burden of full-intensity investments across different-type agents for (roughly) half of the agents from $N_{\theta'}$, for $\theta' \neq \theta$. In particular, we construct a *minimal set of full-intensity investments* across different-type agents which are distributed across the agents in a relatively uniform way.

COROLLARY 2. Assume Assumption 1—Assumption 3, and consider a preference specification u. Then, provided that the capacity constraint is sufficiently loose, with the particular form $R \geq n_A + (n_B-1)/2$, if the level of assortative interests is sufficiently high, with the particular form $\beta \leq [R+(n_B-n_A)]/2(n_A-1)$, then there exists a class of strategy profiles $x \in X$, invariant to any relabelling of the names of the agents, which induces maximally homophilic stable networks $g = g(x) \in S(u)$.

Given the partitions of groups in Observation 1, such a class of strategy profiles x can be described as: (1) each agent $i \in N_A^L$ invests with full intensity $x_{ij} = 1$ in each agent $j \in N_B^L$, and each agent $i \in N_A^H$ invests with full intensity $x_{ij} = 1$ in each agent $j \in N_B^H$, whereas (2) each agent $j \in N_B^L$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^H$, and each agent $j \in N_B^H$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^L$.[21]

[21] Given the partitions of groups in Observation 1, conditions (1) and (2) of Corollary 2 above lead to

Throughout the paper, we present several examples (Example 1–Example 3) to illustrate our main results on stability.

EXAMPLE 1. —*A Maximally Homophilic Network.*

Consider a population $N = \{1,\dots,7\}$ such that $N_A = \{1,2,3,4\}$ and $N_B = \{5,6,7\}$. Notice then that $\alpha(n_A) = \alpha(4) = 2$ and $\alpha(n_B) = \alpha(3) = 1$. Therefore, $R = 5$ by Assumption 3.

Following Observation 1, consider that the group N_A is divided into two subgroups, $N_A^L = \{1,2\}$ and $N_A^H = \{3,4\}$. Similarly, the group N_B is separated into two subgroups, $N_B^L = \{5\}$ and $N_B^H = \{6,7\}$. Let us consider a maximally homophilic network where each agent from group N_A makes full-intensity investments in each of the other three agents in her same subgroup, whereas each agent from group N_B makes full-intensity investments in each of the other two agents in her same subgroup. Using Corollary 2, consider that each agent from the subgroup $N_A^L = \{1,2\}$ makes a full-intensity investment in the (unique) agent from the group $N_B^L = \{5\}$, whereas each agent from the subgroup $N_A^H = \{3,4\}$ makes full-intensity investments in each agent from the subgroup $N_B^H = \{6,7\}$. On the other hand, consider that agent 5 (the unique member of the subgroup N_B^L) makes full-intensity investments in each agent from the subgroup $N_A^H = \{3,4\}$, while each agent from the subgroup $N_B^H = \{6,7\}$ makes full-intensity investments into each agent from the subgroup $N_A^L = \{1,2\}$. Then, it can be easily verified that, for each of the twelve possible pairs $(i,j) \in N_\theta \times N_{\theta'}$ of different-type agents, we have that one agent, either $i \in N_\theta$ or $j \in N_{\theta'}$, invests with full intensity in the other agent. Thus, as required by Lemma 2, for each of the $n(n-1) = 7 \times 6 = 42$ possible pairs of agents in the society, at least one of the two agents makes a full-intensity investment in the other agent. Therefore, no pair of agents have incentives to deviate, complying with condition 2. of Definition 1.

Given the description provided thus far, notice that while agents 3 and 4, belonging to subgroup N_A^H, are exhausting their 5 units of resource, the rest of agents in the N_A^L, N_B^L, and N_B^H, are only inesting 4 units of resource. Thus they still wish to allocate their remaining 1 unit. We complete the description of the strategy profile by considering that (i) each agent $i \in N_A^L = \{1,2\}$ invests $1/2$ units in each agent $j \in N_B^H = \{6,7\}$, (ii) agent 5 invests $1/2$ units in each agent $i \in N_A^L = \{1,2\}$, and (iii) each agent $j \in N_B^H = \{6,7\}$ invests $1/2$ units in each agent $i \in N_A^H = \{3,4\}$. As a result, each link between each pair of agents $(i,j) \in N_A \times N_B$ features (uniform) quality $g_{ij} = 3/4$. We can now derive the ratio $d_i(x)/s_i(x)$ for each agent $i \in N$ as follows:

$$d_i/s_i = (1+5/4)/3 = 3/4 \text{ for } i \in N_A^L;\;\; d_i/s_i = (1+1)/3 = 2/3 \text{ for } i \in N_A^H;$$
$$d_j/s_j = (3/2+1)/2 = 5/4 \text{ for } j \in N_B^L;\;\; d_j/s_j = (3/2+3/2)/2 = 3/2 \text{ for } j \in N_B^H.$$

Therefore, for $\beta \in (0,2/3]$ all agents behave individually as described by the solution [a]

that each agent $i \in N_\theta$ must invest with full intensity in links to either $(n_\theta - 1) + \alpha(n_{\theta'})$ or $(n_\theta - 1) + [n_{\theta'} - \alpha(n_{\theta'})]$ other agents in the population, depending on whether $i \in N_\theta^L$ or $i \in N_\theta^H$. This captures a relatively uniform distribution of efforts by the agents to contribute to the formation of links.

in Fig. 1. We can thus guarantee that the proposed strategy profile, which induces a maximally homophilic network, is immune both against unilateral and bilateral deviations. Indeed, for the particulars of this example, notice the condition on assortative interests stated in Corollary 2 requires that

$$\beta \leq \frac{R+(n_B-n_A)}{2(n_A-1)} = \frac{5+(3-4)}{2(4-1)} = \frac{2}{3}.$$

We turn now to explore extreme forms of heterophilic patterns. Proposition 2 provides conditions that characterize when minimally homophilic networks arise as stable ones. Given a strategy profile $x \in X$, the following lower bound

$$\tilde{\beta}(x) \equiv \sup_{i \in N_\theta,\, \theta \in \Theta} \frac{2n_{\theta'}}{(R-n_{\theta'})+\sum_{j \in N_\theta^i} x_{ji}}$$

on the level of assortative interests of the population will be useful to grasp how assortative interests lead to maximally homophilic networks.

PROPOSITION 2. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x a strategy profile that induces a minimally homophilic friendship network $g = g(x)$. Then, the network g is stable, i.e., $g \in S(u)$, if and only if:

1. *Robustness against unilateral deviations:* the level of assortative interests of the population described by β is sufficiently low, with the particular form given by $\beta \geq \tilde{\beta}(x)$.

2. *Robustness against bilateral deviations:* provided that the resource R is sufficiently large, with the particular form given by $R \geq n_A + (n_B-1)/2$, then for each pair of agents from a common group, $i, j \in N_\theta$, with $i \neq j$, for each type $\theta \in \Theta$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$.

Similarly to Proposition 1, Proposition 2 characterizes investment profiles that lead to minimally homophilic networks, in terms of the primitives n_A, n_B, β, R. We obtain a converse insight to the one provided by Proposition 1. Even in the presence of low assortative interests, a certain degree of quality of the homophilic relations is necessary to sustain minimally homophilic stable networks. In particular, the connection between same-type agents i, j must feature a linkage quality $g_{ij} \geq 1/2$.

Corollary 3 provides a bound on the level of assortative interests that guarantees the existence of only minimally homophilic networks.

COROLLARY 3. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x be a strategy profile that induces a friendship network $g = g(x)$. Provided that $R \geq n_A + (n_B-1)/2$, suppose that x satisfies that for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$. Then, if the level of assortative interests in the population is sufficiently low, with the particular form given by $\beta > \beta_H$, where β_H is the bound on β specified in Eq. (4), then the unique class of stable networks consists entirely of minimally homophilic netwoks.

The sufficient condition stated in Corollary 3 follows by considering a bound β_H on the cutoff values $\hat{\beta}(x)$ such that, if $\beta > \beta_H$, then each agent would choose to invest with full intensity in all different-type others as captured by [c] of Fig. 1. The value of β_H is derived by noticing that the bound $2n_{\theta'}/(R-n_{\theta'})$, corresponds to a situation where an agent $i \in N_\theta$ receives no investments from same-type agents and full investments from all different-type agents, that is $\sum_{j\in N_{\theta'}} x_{ji} = n_{\theta'}$ and $\sum_{j\in N_\theta^i} x_{ji} = 0$. Then, if $\beta > 2n_{\theta'}/(R-n_{\theta'})$, even such an agent would optimally behave as captured by [c] in Fig. 1.

Corollary 4 provides conditions that ensure the existence of stable minimally homophilic networks where the distribution of full-intensity investments across agents is relatively uniform, conditional on their characteristics. It will be useful to set the (type-dependent) integer $l_\theta \equiv \max\{n_\theta - n_{\theta'}, 0\}$.

COROLLARY 4. Assume Assumption 1—Assumption 3, and consider a preference specification u. Then, provided that the total resource R available to the agents is sufficiently large, under the particular requirement that $R \in \{n_A + \alpha(n_B), \ldots, n-2\}$, if the level of assortative interests β satisfies $\beta \geq \beta_h$, where β_h is the bound on β specified in Eq. (4), then there exists a class of strategy profiles $x \in X$, invariant to any relabelling of the names of the agents, which induces minimally homophilic stable networks $g = g(x)$.

In particular, such a class of strategy profiles x can be constructed as follows: for each type $\theta \in \Theta$, (1) upon relabelling the names of the agents in N_θ, set $N_\theta \equiv \{i_1, i_2, \ldots, i_{n_\theta}\}$, (2) for each agent $i_k \in N_\theta$, let then $N_{i_1}(x) = N_{\theta'} \cup \{i_2, \ldots, i_{1+(R-n_A+l_\theta)}\}$, $N_{i_2}(x) = N_{\theta'} \cup \{i_3, \ldots, i_{2+(R-n_A+l_\theta)}\}$, and so on iteratively, until reaching $N_{i_{n_\theta}}(x) = N_{\theta'} \cup \{i_1, \ldots, i_{R-n_A+l_\theta}\}$.[22]

Notice that any profile from the class described in Corollary 4 satisfies the key condition (required by Lemma 2) that $j \notin N_i(x)$ must imply $i \in N_j(x)$, for each pair of different agents $i, j \in N$. Also, the proposed family of profiles entails that each agent $i \in N_A$ invests with full intensity in each of the n_B different-type agents and in $R - n_B$ same-type agents—since, in this case, we have $l_A = \max\{n_A - n_B, 0\} = n_A - n_B$. On the other hand, each agent $i \in N_B$ invests with full intensity in each of the n_A different-type agents and in $R - n_A$ same-type agents—since $l_B = \max\{n_B - n_A, 0\} = 0$. Also, it follows that the agents who belong to the largest group can spare more of their resource to fully invest in same-type agents after investing (with full intensity) in all different-type agents. With this construction for the required minimal set of full-intensity investments the burden of investments across same-type agents is distributed uniformly.

[22] To appreciate better the set of agents who receive full investments by each agent of each subgroup in the corollary, consider that the agents from each set $N_\theta \equiv \{i_1, i_2, \ldots, i_{n_\theta}\}$ are arranged in a circular fashion. Then each agent i_k invests with full intensity in each of the following $i_{k+1}, \ldots, i_{k+(R-n_A+l_\theta)}$ agents. We continue in this way until each of the last agents from list $\{i_1, i_2, \ldots, i_{n_\theta}\}$ invests fully in the subsequent agents until completing full investments in $R - n_A + l_\theta$ agents.

EXAMPLE 2. —*A Minimally Homophilic Network.*

As in Example 1, consider a population $N = \{1, \ldots, 7\}$ such that $N_A = \{1,2,3,4\}$ and $N_B = \{5,6,7\}$. Recall that $\alpha(n_A) = \alpha(4) = 2$ and $\alpha(n_B) = \alpha(3) = 1$, and that $R = 5$

Note first that $l_A = 1$ and $l_B = 0$. Then, using the construction proposed by Corollary 4, we consider a strategy profile x such that: $N_1(x) = N_B \cup \{2,3\}$, $N_2(x) = N_B \cup \{3,4\}$, $N_3(x) = N_B \cup \{4,1\}$, $N_4(x) = N_B \cup \{1,2\}$, $N_5(x) = N_A \cup \{6\}$, $N_6(x) = N_A \cup \{7\}$, and $N_7(x) = N_A \cup \{5\}$. Notice that all agents in the society are exhausting their 5 units of resource. Moreover, it can be easily verified that, for each of the twelve possible pairs $(i, j) \in N_A \times N_A$, $i \neq j$, exactly one agent invests with full intensity in the other agent. Similarly, for each of the six possible pairs $(i, j) \in N_B \times N_B$, $i \neq j$, exactly one agent invests with full intensity in the other agent. As required by Lemma 2, for each of the 42 possible pairs of agents in the society, at least one of the two agents makes a full-intensity investment in the other agent. Therefore, no pair of agents have incentives to deviate, complying with condition 2. of Definition 1. We consider a uniform distribution of investments across same-type agents. As a result each link between each pair of different agents $(i, j) \in N_\theta \times N_\theta$ from a common subgroup N_θ features quality $g_{ij} = 1/2$. The ratio $d_i(x)/s_i(x)$ for each agent $i \in N_A$ is $d_i/s_i = 3/2$, whereas for each agent $j \in N_B$, we have $d_j/s_j = 4/1$. Thus for $\beta \geq 4$ each agent behaves individually as described by solution [c] in Fig. 1. The proposed strategy profile is therefore immune both against unilateral and bilateral deviations. For the particulars of this example, the condition on assortative interests stated in Corollary 4 requires that $\beta \geq \beta_h = n_A/(R - n_A) = 4$.

The results provided by Proposition 3 are quite useful to complement our picture of stable friendship networks. If dissasortative interests prevail, then maximally homophilic networks are not stable. Conversely, societies in which assortative interests prevail do not feature stable minimally homophilic networks.

PROPOSITION 3. Assume Assumption 1—Assumption 3, and consider a preference specification u. Then,

(i) if the interests for making friends β of the society lean towards disassortativity, with the particular form $\beta \in (1, +\infty)$, then there is no stable maximally homophilic network;

(ii) if the interests for making friends β of the society lean towards assortativity, with the particular form $\beta \in (0, 1]$, then there is no stable minimally homophilic network.

Intuitively, under the presence of capacity constraints, (i) if the level of assortative interests is low—so that agents value marginal investments in different-type agents more than in same-type individuals—, then agents choose not to devote the scarce resource to invest with full intensity in all other same-type agents. When group sizes are asymmetric, members of the larger group will be relatively more constrained in this respect because they are required to invest in a relatively higher number of agents under the description of a maximally homophilic network. Similarly, (ii) if the level of assortative interests is high, then agents prefer not to devote the scarce resource to invest with full intensity in all other different-type agents. Members of the smaller group will in this case be relatively more constrained in this respect.

Finally, the insights of Corollary 5 allow us to give a full description of stable friendship networks. In particular, with a flavor similar to that of the results in Corollary 1 and Corollary 3, Corollary 5 provides an interval for the level of assortative interests that guarantees the existence of only partially homophilic networks.

COROLLARY 5. Assume Assumption 1—Assumption 3, and consider a preference specification u. Let x be a strategy profile that induces a friendship network $g = g(x)$. Suppose that x satisfies that for each pair of agents from different groups, $i \in N_A$ and $j \in N_B$, we have $j \notin N_i(x) \Rightarrow i \in N_j(x)$. Then, if the level of assortative interests in the population β is intermediate, with the particular form given by $\beta_l < \beta < \beta_h$, then the unique class of stable networks consists entirely of partially homophilic netwoks.

3.5 Main Takeaways on Stable Patterns

We can combine the results of Proposition 1—Proposition 3 and those of Corollary 1—Corollary 5 to establish the key bounds that describe how homophily levels in stable networks depend on the level of assortative interests in the society. Our analysis has shown that:

1. for values $\beta \in (0, \beta_L)$, *only* maximally homophilic networks *are* stable;
2. for values $\beta \in [\beta_L, \beta_l]$, both maximally homophilic and partially homophilic networks *may be* stable;
3. for values $\beta \in (\beta_l, \beta_h)$, *only* partially homophilic networks *may be* stable;
4. for values $\beta \in [\beta_h, \beta_H]$, minimally homophilic *are* stable and partially homophilic networks *may be* stable;
5. for values $\beta \in (\beta_H, +\infty)$, *only* minimally homophilic *are* stable.

Using the details of Example 1 and Example 2, the bounds on the level of assortative interests in Eq. (4) take values $\beta_L = 1/3$, $\beta_l = 17/24$, $\beta_h = 4$ and $\beta_H = 8$. For those examples we thus know that only maximally homophilic networks are stable for $\beta \in (0, 1/3)$, whereas only minimally homophilic networks are stable for $\beta \in (8, +\infty)$. Only partially homophilic networks are stable for $\beta \in (17/24, 4)$. Also, both maximally homophilic and partially networks may coexist as stable ones for $\beta \in [1/3, 17/24]$, whereas both minimally homophilic and partially homophilic networks may coexist as stable ones for

$\beta \in [4,8]$. Recall that we have described a maximally homophilic network for $\beta \in (0,2/3]$ (Example 1), and a minimally homophilic network for $\beta \in [4,+\infty)$ (Example 2).

We can now be specific about when existence and uniqueness of classes of stable networks can be guaranteed in our setting. Corollary 2 showed that if the level of assortative interests is relatively high, with the particular form $\beta \leq [R+(n_B-n_A)]/2(n_A-1)$, then there exists a strategy profile x that induces a class of stable maximally homophilic networks $g = g(x)$. Since β_L can be written as $[R+(1-n_A)]/2(n_A-1)$, it can be directly noted that $\beta_L < [R+(n_B-n_A)]/2(n_A-1)$ in our setting. Therefore, existence of stable patterns are guaranteed for assortative interests $\beta \leq \beta_L$. In addition, Corollary 4 granted that if the level of assortative interests is relatively low, with the particular form $\beta \geq \beta_h$, then there exists a strategy profile x that induces a class of stable minimally homophilic networks $g = g(x)$. The implications on existence and uniqueness follow them by combining the implications of Corollary 1 and Corollary 2 (on the side of extreme homophilic patters), and of and Corollary 3 and Corollary 4 (on the side of extreme heterophilic patterns).

Our results convey the natural message that resulting homophily levels in friendship networks are positively related with the assortative interests of the individuals in the society. However, our insights on (potentially) beneficial bilateral deviations (Lemma 2)—via the premium of mutual efforts—also show that links between each pair of agents with different characteristics must also be sponsored with full intensity by (at least) one of the two friends in order to sustain extreme forms of homophilic patterns. This insight that homophily does not arise in a way fully isolated (from heterophilic links of a certain quality) is also consistent with the non-trivial results that, for assortative levels $\beta \in [\beta_L, \beta_l]$, maximally homophilic networks coexist with partially homophilic ones. Such messages also extend to our investigation of extreme forms of stable heterophilic patterns. We observe that strong forms of heterophilic patterns cannot arise unless certain quality levels of homophilic connections are also present. In consonance with such messages, note also that, for assortative levels $\beta \in [\beta_h, \beta_H]$, both minimally and partially homophilic networks coexist. Finally, note that our model delivers the insight that extreme forms of homophilic and heterophilic patterns cannot coexist simultaneously under a common level of assortative interests.

As to the role played by the relative sizes of the two population groups, N_A and N_B, it can be shown that β_l decreases as n_A increases, when keeping n_B and R fixed. In this case, β_L decreases, while the difference $n_A - n_B$ of the two population sizes rises. As a consequence, when the size of the larger group leads to high discrepancies between group sizes, it becomes harder to sustain maximally homophilic networks as stable ones. This insight contrasts the cases of societies that feature similar sizes for their groups, for which it is easer to sustain maximally homophilic networks as stable ones. Such messages are quite consistent with some results of the empirical analysis conducted by Currarini et al. (2009). In their data, it is precisely the presence of large discrepancies between group sizes what causes connections to fail to give extreme homophily patterns in the entire student population. Thus, this empirically obtained message goes in the same direction as our results on stability of maximally homophilic patterns when groups are very different in their sizes.

We turn now to explore a particular class of partially homophilic networks that can be stable for "intermediate" assortative levels $\beta \in [\beta_L, \beta_H]$.

3.6 A Class of Partially Homophilic Networks

An interesting special case of partially homophilic networks is that in which *all* agents behave unilaterally as described by the interior solution [b] in Fig. 1. Thus, no agent invests with full intensity neither in all their same-type fellows nor in all the different-type agents. In general, though, it turns out difficult to guarantee the existence of such partially homophilic networks as stable ones. For the particular case where both population groups have a common even size, we provide a method, in Observation 2 below, to construct a family of strategy profiles that induce stable partially homophilic networks with the above mentioned feature. In addition, our proposal seeks to distribute as uniformly as possible the burden of full-intensity investments across the agents in the population.

OBSERVATION 2. We restrict attention to those populations such that $n_A = n_B = n/2$ for $n/2$ even. Upon relabelling the names of the agents in the population, let us set $N_A \equiv \{i_1, i_2, \ldots i_{n_A}\}$ and $N_B \equiv \{j_1, j_2, \ldots j_{n_B}\}$. Consider that the agents from each of the two lists $\{i_1, i_2, \ldots i_{n_A}\}$ and $\{j_1, j_2, \ldots j_{n_B}\}$ are arranged in a circular fashion. In addition, exactly as proposed in Observation 1, let us consider a partition of each of the two population groups N_θ, for $\theta \in \Theta$, into two sets, N_θ^L and N_θ^H, according to: (a) N_A is partitioned into $N_A^L = \{i_1, \ldots, i_{\alpha(n_A)}\}$ and $N_A^H = \{i_{\alpha(n_A)} + 1, \ldots, i_{n_A}\}$, whereas (b) N_B is partitioned into $N_B^L = \{j_1, \ldots, j_{\alpha(n_B)}\}$ and $N_B^H = \{j_{\alpha(n_B)} + 1, \ldots, j_{n_B}\}$.

Given those ingredients, the suggested method consists of two steps. In the first step, we describe the minimal set of full-intensity investments which guarantees that no pair of agents have bilateral incentives to deviate (as required by Lemma 2). The second step describes how agents invest in the remaining agents. The underlying logic of this step is that the remaining investments are such that each agent exhausts her available resource while, at the same time, the induced profile is such that each agent behaves unilaterally as the aforementioned solution [b] in Lemma 1.

First Step.— In regard to same-type fellows, consider that, for each type $\theta \in \Theta$, each agent $i \in N_\theta$ invests with full intensity in the subsequent $\alpha(n_\theta)$ agents from the same-type list following the suggested circular arrangement. As to how agents invest with full intensity in different-type agents, consider that (a) each agent $i \in N_A^L$ invests $x_{ij} = 1$ in each agent $j \in N_B^L$; (b) each agent $i \in N_A^H$ invests $x_{ij} = 1$ in each agent $j \in N_B^H$; (c) each agent $j \in N_B^L$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^H$, and (d) each agent $j \in N_B^H$ invests with full intensity $x_{ji} = 1$ in each agent $i \in N_A^L$.

The description provided in this first step already guarantees the condition required by Lemma 2 to prevent profitable bilateral deviations from the profile x. In particular, all agents invest with full intensity in $\alpha(n_A) + \alpha(n_B) = \alpha(n)$ other agents. The described full-intensity investments in the subsequent $\alpha(n_\theta)$ same-type agents (along the circular arrangement) ensure that, for each pair of same-type agents, at least one of them is investing with full intensity in the other agent. In addition, the crossed-investments among different type-agents suggested simply replicate the description proposed in Observation 1 to guarantee the robustness against bilateral deviations of the class of profiles described in Corollary 2. Corollary 2 showed that such cross-investments involving the four population subgroups ensured that, for each pair of different-type agents, at least one of them invests with fully intensity into the other.

Note that our description thus far entails that each agent behaves unilaterally as described by the interior solution [b] in Fig. 1. We still need to describe the pending investments so that each agent exhausts her resource. Let $\hat{N}_i(x)$ be the minimal set of full-investments of agent i constructed as suggested above. In general, we have $\hat{N}_i(x) \subseteq N_i(x)$, though it could that such an inclusion relationship holds strictly in some particular cases.

*Second Step.—*Let us reconsider the condition over total qualities $d_i/s_i = \beta$, which is required for agent i to makes the optimal (unilateral) investment choice described by [b] in Fig. 1. Given our description of the first step, such a condition can be rewritten as

$$\frac{|\hat{N}_i(x) \cap N_{\theta'}| + |\{j \in N_{\theta'} \mid i \in \hat{N}_j(x)\}| + \sum_{j \in N_{\theta'} \setminus \hat{N}_i(x)} x_{ij} + \sum_{\{j \in N_{\theta'} \mid i \notin \hat{N}_j(x)\}} x_{ji}}{|\hat{N}_i(x) \cap N^i_\theta| + |\{j \in N^i_\theta \mid i \in \hat{N}_j(x)\}| + \sum_{j \in N^i_\theta \setminus \hat{N}_i(x)} x_{ij} + \sum_{\{j \in N^i_\theta \mid i \notin \hat{N}_j(x)\}} x_{ji}} = \beta. \quad (5)$$

In addition to the requirements in Eq. (5), we must also ensure that that each agent $i \in N$ exhausts her available resource, that is

$$\sum_{j \notin \hat{N}_i(x)} x_{ij} = R - |\hat{N}_i(x)|. \quad (6)$$

In Example 3 we construct partially homophilic network by using the method in Observation 2.

EXAMPLE 3. —*A Partially Homophilic Network.* Consider a population consisting of eight agents such that half of them have one characteristic or the other. Thus, we have $N = \{1,2,3,4\} \cup \{5,6,7,8\}$, with $N_A = \{1,2,3,4\}$ and $N_B = \{5,6,7,8\}$. Let $R = 5$. Notice that $\alpha(n) = \alpha(8) = 8/2 = 4$, and $\alpha(n_A) = \alpha(n_B) = \alpha(4) = 4/2 = 2$.

By resorting to the partitions of each population group described in the first step of Observation 2, consider that the group N_A is divided into two subgroups, $N^L_A = \{1,2\}$ and $N^H_A = \{3,4\}$. Similarly, the group N_B is separated into two subgroups, $N^L_B = \{5,6\}$ and $N^H_B = \{7,8\}$. Then, the class of investment profiles described in Observation 2 requires that we set $\hat{N}_1(x) = \{2,3\} \cup \{5,6\}$, $\hat{N}_2(x) = \{3,4\} \cup \{5,6\}$, $\hat{N}_3(x) = \{4,1\} \cup \{7,8\}$, $\hat{N}_4(x) = \{1,2\} \cup \{7,8\}$, $\hat{N}_5(x) = \{6,7\} \cup \{3,4\}$, $\hat{N}_6(x) = \{7,8\} \cup \{3,4\}$, $\hat{N}_7(x) = \{8,5\} \cup$

$\{1,2\}$, and $\hat{N}_8(x) = \{5,6\} \cup \{1,2\}$. Up to here no agent is investing with full intensity neither in all of the remaining same-type agents, nor in all of the different-type agents, as required in the class of partially homophilic networks that we are considering. Now, using the condition in Eq. (5) of the second step of Observation 2 for a level of assortative interest β, we need to consider

$$\begin{aligned}\beta &= \frac{4+x_{17}+x_{18}+x_{51}+x_{61}}{4+x_{14}+x_{21}} = \frac{4+x_{27}+x_{28}+x_{52}+x_{62}}{4+x_{21}+x_{32}} \\ &= \frac{4+x_{35}+x_{36}+x_{73}+x_{83}}{4+x_{32}+x_{43}} = \frac{4+x_{45}+x_{46}+x_{74}+x_{84}}{4+x_{43}+x_{14}} \\ &= \frac{4+x_{51}+x_{52}+x_{35}+x_{45}}{4+x_{58}+x_{65}} = \frac{4+x_{61}+x_{62}+x_{36}+x_{46}}{4+x_{65}+x_{76}} \\ &= \frac{4+x_{73}+x_{74}+x_{17}+x_{27}}{4+x_{76}+x_{87}} = \frac{4+x_{83}+x_{84}+x_{18}+x_{28}}{4+x_{87}+x_{58}}.\end{aligned}$$

In addition, Eq. (6) requires that we add the constraints

$$\begin{aligned}&x_{14}+x_{17}+x_{18}=1, x_{21}+x_{27}+x_{28}=1, x_{32}+x_{35}+x_{36}=1, x_{43}+x_{45}+x_{46}=1,\\ &x_{58}+x_{51}+x_{52}=1, x_{65}+x_{61}+x_{62}=1, x_{76}+x_{73}+x_{74}=1, x_{87}+x_{83}+x_{84}=1.\end{aligned}$$

Given the main takeaways of the model in Subsection 3.5 for values $\beta \in [\beta_L, \beta_H] = [1/3, 8]$ we ensure that the so derived network $g = g(x)$ is robust against unilateral deviations. For the particular value $\beta = 8/7$, we can then propose symmetric non full-intensity investments so that for each agent $i \in N$, we have $x_{ij} = 1/3$ for each $j \notin \hat{N}_i(x)$.

4 Efficiency of Friendship Networks

Our analysis of efficiency properties relies on a classical *utilitarian* approach where the social planner gives all agents the same importance, regardless of their identities and characteristics.[23] In particular, we assume that the *(social) value of friendship networks* is described by a function $v : G \to \mathbb{R}_+$, specified as

$$v(g(x)) \equiv \sum_{i \in N} \pi_i(g(x)). \tag{7}$$

The notion of efficiency that we use follows closely Jackson and Wolinsky (1996). In addition, we naturally require the social planner to face the same capacity constraints that restrict the agents' choices. Formally,

[23] Utilitarian approaches have been commonly pursued in literature that explores the relationship between stable and efficient networks. See, among others, Jackson and Wolinsky (1996), Calvó-Armengol (2003), Goyal and Vega-Redondo (2007), Bloch and Jackson (2007), and, Bloch and Dutta (2009).

DEFINITION 4. A friendship network $\hat{g} = g(\hat{x})$ induced by an investment profile $\hat{x}$ is efficient if, conditional on considering investment profiles that satisfy the capacity constraints, the investment profile $\hat{x}$ maximizes the sum of the utilities of all the agents in the population, that is, if $v(\hat{g}(\hat{x})) \geq v(g(x))$ for $\hat{x} \in X$ and for each $x \in X$.

Since the social planner seeks to maximize the sum of the agents' utilities, we consider the class of friendship networks where all the agents exhaust the available resource R.

Proposition 4 shows that any efficient pattern must necessarily have common resulting qualities of both same-type and different-type friendship links across all individuals within each of the two population groups. The key insight provided by Proposition 4 exploits the assumptions that preferences are common across agents and that they are (strictly) convex in the (s_i, d_i) space (Assumption 2–(3)). The logic of the result in Proposition 4 relies on the implication that for each feasible investment profile $x \in X$, we can find another feasible profile $\hat{x} \in X$—which can be related to x in a precise way—such that: (i) the same-type $s_i(\hat{x})$ and different-type $d_i(\hat{x})$ qualities are constant across all agents within each population group N_A and N_B, and (ii) the social value derived from $\hat{x}$ is no less than the one derived from x. Importantly, it also follows that $v(g(\hat{x})) > v(g(x))$ unless the profile x features also common qualities $s_i(x)$ and $d_i(x)$ across all agents within each population group.

PROPOSITION 4. Assume Assumption 2 and Assumption 3, and consider a preference specification u. Let $\hat{x}$ be an investment profile that induces an efficient network $\hat{g} = g(\hat{x})$. Then, the total qualities $(s_i(\hat{x}), d_i(\hat{x}))$ must be common across all agents in each of the two population groups, that is, $s_i(\hat{x}) = s_\theta(\hat{x})$ and $d_i(\hat{x}) = d_\theta(\hat{x})$ for each agent $i \in N_\theta$ and each type $\theta \in \Theta$.

Proposition 4 enables us to restrict attention to a particular family of investment profiles $\hat{x}$ that are the only candidates to induce an efficient network. Heuristically, as can be noted from the proof of Proposition 4, such a family of profiles $\hat{x}$ is characterized by the following proposal of aggregate investments. For each agent $i \in N_\theta$ and each type $\theta \in \Theta$, let

(a) $\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = y_{\theta\theta}$, and

(b) $\sum_{j \in N_{\theta'}} x_{ij} = y_{\theta\theta'}$ and $\sum_{j \in N_{\theta'}} x_{ji} = z_{\theta\theta'}$.

The family of profiles $\hat{x}$ specified above is the unique family able to induce qualities $(s_i(\hat{x}), d_i(\hat{x}))$ that be constant across all agents within each population group. It follows that $s_i(\hat{x}) = y_{\theta\theta}$ and $d_i(\hat{x}) = (1/2)[y_{\theta\theta'} + z_{\theta\theta'}]$ for each $i \in N_\theta$ and each $\theta \in \Theta$. For such a family of profiles, we must consider the capacity constraints imposed on the agents (Assumption 3) with equality, so that $y_{\theta\theta} + y_{\theta\theta'} = R$ for each type $\theta \in \Theta$ and for the type $\theta' \neq \theta$. Finally, as also indicated in the proof of Proposition 4, such aggregate investments must also satisfy $n_A z_{AB} = n_B y_{BA}$ and, similarly, $n_B z_{BA} = n_A y_{AB}$.[24] By putting together all

[24] This is obtained by equalizing the aggregate investments that all agents from a group N_θ make in

the considerations above, we are left with a tractable description of the problem that faces the social planner, namely, choosing profiles $\hat{x}$ in order to maximize the expression

$$\begin{aligned} v(g(\hat{x})) = & n_A u\big(y_{AA}, (1/2n_A)(nR - n_A y_{AA} - n_B y_{BB})\big) \\ & + n_B u\big(y_{BB}, (1/2n_B)(nR - n_A y_{AA} - n_B y_{BB})\big). \end{aligned} \tag{8}$$

Given all the ingredients above, we derive sufficient conditions, in Proposition 5, that characterize unique classes of investment profiles that induce efficient networks. Each class is described by the above mentioned aggregate outgoing/incoming investments $y_{\theta\theta}$, for each $\theta \in \Theta$. Nevertheless, note that each class includes multiple profiles $\hat{x}$ because the derived conditions do not depend on the particular investments $\hat{x}_{ij}$ from each agent i to another agent j in her same group.

PROPOSITION 5. Assume Assumption 2 and Assumption 3, and consider a preference specification u. Let $\hat{x}$ be an investment strategy profile that satisfies the necessary condition given by Proposition 4. Then,

(i) if the level of assortative interests in the population is sufficiently high, with the particular form given by $\beta < \beta_l$, then the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies

$$\begin{aligned} &\text{for } i \in N_\theta, \quad \sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = n_\theta - 1, \quad \sum_{j \in N_{\theta'}} x_{ij} = R - (n_\theta - 1), \text{ and} \\ &\text{for } i \in N_A, \quad \sum_{j \in N_B} x_{ji} = (n_B/n_A)[R - (n_A - 1) + (n_A - n_B)]; \\ &\text{for } i \in N_B, \quad \sum_{j \in N_A} x_{ji} = (n_A/n_B)(R - (n_A - 1)). \end{aligned}$$

(ii) if the level of assortative interests in the population is sufficiently low, with the particular form given by $\beta > \beta_h$, then the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies, for each $i \in N_\theta$ and each $\theta \in \Theta$, $\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = R - n_{\theta'}$, $\sum_{j \in N_{\theta'}} x_{ij} = n_{\theta'}$, and $\sum_{j \in N_{\theta'}} x_{ji} = n_{\theta'}$.

The class of investment profiles $\hat{x}$ identified in (i) of Proposition 5 correspond to maximally homophilic networks and for agents $i \in N_\theta$ give us common qualities

$$s_i(\hat{x}) = n_\theta - 1 \text{ and } d_i(\hat{x}) = \frac{n(R - (n_A - 1)) + n_B(n_A - n_B)}{2n_\theta}.$$

The profiles identified in (ii) of Proposition 5 correspond to minimally homophilic network and deliver common qualities $s_i(\hat{x}) = R - n_{\theta'}$ and $d_i(\hat{x}) = n_{\theta'}$ for all $i \in N_\theta$.

all agents from the other group $N_{\theta'}$ to the aggregate investments that the agents from $N_{\theta'}$ receive from all agents from N_θ. Also notice that these requirements imply that $z_{\theta\theta'} = \sum_{j \in N_{\theta'}} x_{ji} = (n_{\theta'}/n_\theta)[R - y_{\theta'\theta'}]$.

Thus, only maximally homophilic networks are efficient for assortative levels $\beta < \beta_l$, whereas only minimally homophilic networks are efficient for values $\beta > \beta_h$.

A few insights emerge from Proposition 4 and Proposition 5. We observe first a certain discrepancy between stability and efficiency of partially homophilic networks. In particular, although partially homophilic networks may be stable for assortative interests $\beta \in [\beta_L, \beta_l] \cup [\beta_h, \beta_H]$, such networks are not efficient. Secondly, notice that the maximally homophilic network constructed in Example 1 does not satisfy the necessary condition that the resulting aggregate qualities (s_i, d_i) be common across all agents within each population group. In that example, we indeed derived

$$d_i = 9/4 \text{ for } i \in N_A^L \text{ whereas } d_i = 2 \text{ for } i \in N_A^H, \text{ and}$$
$$d_j = 5/2 \text{ for } j \in N_B^L \text{ whereas } d_j = 3 \text{ for } j \in N_B^H.$$

Note that by (i) in Proposition 5, in maximally homophilic efficient networks each agent $i \in N_B$ receives an aggregate investment $\sum_{j \in N_A} x_{ji} = (n_A/n_B)(R - (n_A - 1)) \geq 2$, with strict inequality if $n_A > n_B$, from the agents from the group N_A. Therefore, we obtain the property that for a maximally homophilic network to be efficient if populations sizes are not equal, at least three agents from the larger group N_A must invest a positive amount into each agent from the smaller group N_B. This property is not satisfied in Example 1 since agent $5 \in N_B$ is receiving investments from only two agents from the group N_A—i.e., those agents in $N_A^L = \{1, 2\}$.

The inefficiency of the maximally homophilic network of Example 1 highlights a more general feature of maximally homophilic networks when the population groups differ in their sizes. Conditional on the agents of each group having invested with full intensity in all their same-type fellows, stability requires that at least one agent within each pair of different-type agents invest with full intensity in the other agent. Attaining such minimum full-intensity investments naturally entails asymmetries in investments between the two different groups when their sizes are different. Then, when the two groups differ in their sizes, the efficiency requirement that the aggregate qualities d_i be common across all agents from each group become harder to achieve. When the two groups have the same size, however, there are no asymmetries between the investments between different groups required to attain stability of a maximally homophilic network. Observation 3 gives a method to construct maximally homophilic networks that are simultaneously stable and efficient, provided that the two groups of agents have the same size.

OBSERVATION 3. Consider a situation where $n_A = n_B$. Suppose that the level of assortative interests is sufficiently high, with the particular form $\beta < (R - (n_A - 1))/(n_A - 1) = \beta_l$.

Upon relabelling the names of the agents in the two population groups, let us set $N_A \equiv \{i_1, i_2, \ldots, i_{n_A}\}$ and $N_B \equiv \{j_1, j_2, \ldots, j_{n_B}\}$. Consider a class of strategy profiles x described as follows. For each agent $i_k \in N_A$, let

$N_{i_1}(x) = N_A^{i_1} \cup \{j_2, \ldots, j_{1+(R-(n_A-1))}\}$,

$N_{i_2}(x) = N_A^{i_2} \cup \{j_3, \ldots, j_{2+(R-(n_A-1))}\}$,

and so on iteratively, until reaching

$N_{i_{n_A}}(x) = N_A^{i_{n_A}} \cup \{j_1, \ldots, j_{R-(n_A-1)}\}$.

Analogously, for each agent $j_k \in N_B$, let

$N_{j_1}(x) = N_B^{j_1} \cup \{i_1, \ldots, i_{R-(n_A-1)}\}$,

$N_{j_2}(x) = N_B^{j_2} \cup \{i_2, \ldots, i_{1+(R-(n_A-1))}\}$,

and so on iteratively, until reaching

$N_{j_{n_B}}(x) = N_B^{j_{n_B}} \cup \{i_{n_A}, \ldots, i_R\}$.

Under this proposal, each agent of type $\theta = A$ invests with full intensity in exactly $R-(n_A-1)$ agents of type $\theta = B$ and receives $R-(n_A-1)$ units from the agents of type $\theta = B$. Notice that the available resource $R \in \{n_A + \alpha(n_A), \ldots, n-2\}$, is sufficiently large to allow for each pair of different-type agents to have at least one of them investing with full intensity into the other agent. Thus, the class of described strategy profiles x satisfies the key condition in Lemma 2. Therefore any induced network $g = g(x)$ is robust to bilateral deviations. Under the considered condition on the level of assortative interests $\beta \leq \beta_l$, the networks in the induced class are also robust against unilateral deviations. Furthermore, we obtain the resulting qualities $s_i(x) = n_\theta - 1$ and $d_i(x) = R - (n_\theta - 1)$ for each agent of type $i \in N_\theta$, for each type $\theta \in \Theta$. Any network constructed in this way satisfies the necessary condition for efficiency required by Proposition 4 of common resulting qualities (s_i, d_i) within each population group. For $\beta < \beta_l$ the class of constructed networks satisfies also the sufficient condition for efficiency in (i) of Proposition 5.

For the case of minimally homophilic networks, the stability condition that at least one agent of possible each pair invests fully in the other agent needs to be satisfied *within each group*. This requirement contrasts sharply with what is needed for the case of maximally homophilic networks. In particular, this requirement entails no asymmetries in investments within each group, even when the groups differ greatly in their sizes. In such cases, finding a minimally homophilic network that be simultaneously stable and efficient is always guaranteed, as detailed in Observation 4. In fact, the class of minimally homophilic networks constructed in Example 2 satisfies the necessary condition required by Proposition 4. Furthermore, using the details of this example, we can verify that $\beta_h = 4$. Those minimally homophilic networks suggested in the example were stable for values of the assortative interests $\beta \geq 4$. Thus, the sufficient condition given by Proposition 5 guarantees that the class of minimally homophilic networks constructed in Example 2 are efficient. Furthermore, for $\beta > 4$, the investment profile used to construct the network in the example belongs to the unique class of profiles that induce efficient networks.

OBSERVATION 4. Suppose that the level of assortative interests in the population is sufficiently low, with the particular form $\beta \geq n_A/(R-n_A) = \beta_h$. Let us resort to the class of minimally homophilic networks constructed in Corollary 4. First, upon relabelling the names of the agents in N_θ, for each type $\theta \in \Theta$ and the type $\theta' \neq \theta$, let us set $N_\theta \equiv \{i_1, i_2, \ldots, i_{n_\theta}\}$. Then, for each agent $i_k \in N_\theta$, let us consider $N_{i_1}(x) = N_{\theta'} \cup \{i_2, \ldots, i_{R-(n_A-1)}\}$, $N_{i_2}(x) = N_{\theta'} \cup \{i_3, \ldots, i_{R-(n_A-2)}\}$, and so on iteratively, until reaching $N_{i_{n_\theta}}(x) = N_{\theta'} \cup \{i_1, \ldots, i_{R-n_A}\}$.

Regarding stability, note that the available resource $R \in \{n_A + \alpha(n_B), \ldots, n-2\}$, is sufficiently large to allow for each pair of same-type agents to enjoy a full-investment made by (at least) one of the two agents in the pair. Any network in the suggested class is therefore robust to bilateral deviations. Moreover, while each agent $i \in N_\theta$ is investing exactly $R - n_{\theta'}$ units in her same-type fellows, she is also receiving exactly $R - n_{\theta'}$ units of investment from the agents in her own population group. Thus, for the proposed level of assortative interest $\beta \geq \beta_h$, the suggeted networks are also robust to unilateral deviations.

Regarding efficiency, notice that for each agent $i \in N_\theta$ and each type $\theta \in \Theta$, it follows that the quality of her different-type links is $d_i(x) = n_{\theta'}$, for $\theta' \neq \theta$, while the quality of her same-type links is $s_i = R - n_{\theta'}$. The networks in this class satisfy then the necessary condition for efficiency in Proposition 4 that the qualities (s_i, d_i) be common across all agents within each population group. Furthermore, for $\beta \geq \beta_h$, the suggested class of satisfies also the sufficient condition in (ii) of Proposition 5. The proposed class of mimimally homophilic networks are thus stable and efficient.

For the particular case where the sizes of both population groups are the same, the following corollary to Proposition 5 characterizes efficient networks in terms of the assortative interests of the population.

COROLLARY 6. Assume Assumption 2 and Assumption 3, and consider a preference specification u. Then,

(i) the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies, for each $i \in N_\theta$ and each $\theta \in \Theta$,

$$\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = n/2 - 1, \quad \sum_{j \in N_{\theta'}} x_{ij} = R - (n/2 - 1), \text{ and } \sum_{j \in N_\theta} x_{ji} = R - (n_A - 1)$$

if and only if level of assortative interests in the population is sufficiently high, with the particular form given by $\beta < \beta_l$;

(ii) the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies, for each $i \in N_\theta$ and each $\theta \in \Theta$, $\sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = R - n/2$, and $\sum_{j \in N_{\theta'}} x_{ij} = \sum_{j \in N_{\theta'}} x_{ji} = n/2$ if and only if the level of assortative interests in the population is sufficiently low, with the particular form given by $\beta > \beta_h$;

(iii) if the level of assortative interests in the population is intermediate, with the particular form given by $\beta \in (\beta_l, \beta_h)$, then the investment profile $\hat{x}$ that induces a unique class of efficient networks $g = g(\hat{x})$ satisfies $\hat{y} = y_{AA} = y_{BB}$ with $\beta = (R - \hat{y})/\hat{y} = (n + 2m)/2\hat{y} - 1$ and, therefore, $\hat{y} = R/(1+\beta) = (n + 2(R - (n_A - 1)))/2(1+\beta)$ for an aggregate investment choice $\hat{y} \in (R - n/2, n/2 - 1)$.

Going back to our examples, recall that the stable partially homophilic network constructed in Example 3 required that $\beta = 8/7$. Now, it can be also verified that the network obtained in the example features $\hat{y} = \sum_{j \in N_\theta^i} x_{ij} = \sum_{j \in N_\theta^i} x_{ji} = 7/3$ for each agent $i \in N_\theta$ and each type $\theta \in \Theta$. We observe from the implication in (iii) of Corollary 6 that efficiency requires in this case that $\hat{y} = R/(1+\beta) = 5(1 + 8/7) = 7/3$. Hence, such a partially homophilic network is both stable and efficient.

5 Literature Connections

To capture relations where mutual consent is required to form links, our stability notion builds closely upon the *weak bilateral equilibrium (wBE)* stability concept proposed by Boucher (2015). In turn, such a wBE notion weakens the concept of bilateral equilibrium due to Goyal and Vega-Redondo (2007). Our approach to analyze efficient friendship networks follows the canonical framework proposed by Jackson and Wolinsky (1996).

On the instrumental side, our proposal where agents make continuous-investment choices to build up link qualities can also be found in Bloch and Dutta (2009). Another similarity with Bloch and Dutta (2009) lies in considering a fixed amount of a resource that the agents can allocate in their link formation efforts. Their model is quite different, though, in the sort of questions studied. In particular, they do not consider agents with different characteristics and, accordingly, they do not explore questions of homophily homophily.[25] Another paper in which the agents are exogenously constrained in their capacities to form links is Staudigl and Weidenholzer (2014). Their research questions and approach are quite different from ours as their analysis in not concerned with questions of homophily in populations with agents of different characteristics. Following an evolutionary approach, their investigation focuses on constrained link formation under the stability notion proposed by Bala and Goyal (2000), which considers robustness only against unilateral deviations.

Models where agents have different characteristics abound in the social and economic networks literature. Most efforts have traditionally focused on the question of how assortative interests influence the outcomes of relevant network-based phenomena, such as decisions in labor markets (Montgomery, 1991), opinion formation (Golub and Jackson, 2012; Jimenez-Martinez, 2015; Melguizo-Lopez, 2019), friendship formation via match-

[25] The pioneering contributions on strategic link formation within the economic and social networks literature for the case where agents are not distinguished according to (extrinsic) characteristics are Jackson and Wolinsky (1996) and Bala and Goyal (2000). Other contemporary efforts include Goyal and Vega-Redondo (2007), Hagenbach and Koessler (2010), Galeotti et al. (2013), and Baumann (2021).

ing (Currarini et al., 2009), formation of random networks (Bramoullé et al., 2012), or strategic network formation (De Marti and Zenou, 2017; Iijima and Kamada, 2017).

Perhaps the closest paper to ours in terms of the type of questions asked is Currarini et al. (2009) which proposes a search model of endogenous matching to explore friendship connections. As in our model, in their setting agents care ultimately only about same-type and different-type links. Their exercise is quite different from ours as their goal is to match (and rationalize) certain empirical regularities regarding *only homophilic tendencies*. We attempt to provide a theoretical framework, in which any plausible assortative interests are taken as a primitive, that helps us understand properties of patterns with either homophilic or heterophilic features in the presence of capacity constraints. At the modeling level, we use a simultaneous-move network formation game, while their model is one of dynamic matching. Also within the empirical literature, Mele (2017) proposes a model of network formation in which, as in our setting, agents are divided into two different categories. For the case of sufficiently large networks, his analysis provides useful identification and estimation techniques. Another paper related to ours is De Marti and Zenou (2017), which adapts the symmetric connections model by Jackson and Wolinsky (1996) to a setting in which individuals may have two types and linking costs are endogenous. Unlike our setup, link formation is done through a discrete choice in their model. This choice precludes the critical analysis that our model provides about effort intensities in link quality formation. In particular, our key insights about the presence of heterophilic features in highly homophilic patterns depend crucially on modeling link formation by means of a continuous choice.[26] Regarding efficiency, the results in De Marti and Zenou (2017) are restricted to the comparison of two particular network structures, while our aim is to offer a more general message regarding general properties that the networks must satisfy to be efficient. Another recent paper, quite different from ours in terms of the questions asked and the setup proposed, in which agents differ ex ante in their characteristics is Galeotti et al. (2006).

Baccara and Yariv (2013) consider a model in which homophily arises endogenously as a consequence of a (binary) project choice. Similarly to our model, a given parameter determines the (exogenous) inclination of the agents for one project or another. In their model, stability requires that agents connect sufficiently with (relatively) similar individuals. Baccara and Yariv (2013) provide conditions under which connections between dissimilar agents arise in stable patterns. In addition, for an application in which the projects allow for information sharing, their analysis conveys the message that segregation is easier to maintain when the preferences of the individuals between the two projects are sufficiently opposed. Although their model is quite different from ours, their perspective of studying endogenous homophily levels that may arise from quite general (exogenous) tastes resembles our approach to the topic.

Using a model of dynamic network formation (which incorporates random matching

[26] Also, De Marti and Zenou (2017) consider the stability notion of pairwise stability, which would lead to a profound multiplicity of stable patterns if applied to our setting.

as well), König et al. (2010) consider that agents are restricted by capacity constraints. In their model, the inclusion of capacity constraints is crucial to switch from disassortative to more assortative networks. Their notion of assortativity, however, is quite different from the one considered in our paper (which follows the prevalent notion in the sociology literature). In particular, their approach does not consider relations between individuals with different (extrinsic) characteristics. Instead, their notion of assortativity refers to individuals being prone to connect with others that have similar degrees. In this sense, individuals have a certain (endogenous) characteristic, defined as the number of their (direct) neighbors, and assortativity is interpreted as individuals being relatively more inclined to link with others that have similar numbers of neighbors.

Some of our messages relative to the stability of heterophilic friendship patterns are reminiscent of the insights provided by Galenianos (2021). His model is quite different from ours as he does not consider general friendship connections but focuses on the formation of referral networks in job markets. As a consequence, the motivations of the agents to form links are very specific to job market situations. In particular, workers form links in order to refer to and be refereed by according to the demands of firms. Interestingly, referral networks in his setup feature high levels of heterophily, with the particular form of being hierarchical.[27] Finally, clear reminiscences to our insight that links are sponsored by just one friend while the other free rides can also be found in Galeotti and Goyal (2010). While such an insight depends crucially on the presence of capacity constraints in our model, in their analysis it is the assumption that agents can invest both in acquiring information and in forming links what leads to the implication. Under certain conditions, Galeotti and Goyal (2010) obtain that just a few agents invest in acquiring information, while most agents take advantage of this and invest in linking to the former individuals. Their insight is quite relevant in the formation of friendship links and points towards a key implication that we put forward in the current paper. The underlying mechanisms behind the two qualitatively similar implications are quite different though.

6 Concluding Remarks

This paper has developed a framework to explore stability and efficiency properties of friendship networks in populations of agents with different characteristics. We have taken any plausible underlying level of assortative interests as a primitive of the model. Additionally, we have assumed that investments in each single relationship are bounded and that the agents are capacity-constrained in the amounts of investments they can make relative to the rest of the population. The proposed setting has the flavor of traditional (static) consumption/production choice models. The decision choice that faces each agent when assessing her unilateral incentives resembles a classical utility-maximization problem,

[27] Recent empirical work on labor markets (Hensvik and Skans, 2016; Beaman et al., 2018) offers findings very consistent with such results of hierarchical networks of referrals.

though the feasibility constraint has different fundamentals and form. The presence of capacity constraints stands out as crucial consideration. In those elements, our proposal is perhaps quite different from most available models in the literature on social networks. Our results complement some views offered by the recent papers that deal with homophily and segregation in groups but that do not consider how assortative interest generally vary. We should emphasize that our results do not depend on the parametric specifications of preferences (other than the level of assortative interest), but on the available resource and on the sizes of the groups. This allows us to draw conclusions on how these main ingredients determine the emergence of stable and efficient networks for a broad class of preferences.

We turn now to discuss certain points in which our insights stand in consonance with key implications of certain pieces of available evidence and other theoretical models. First, relative to our welfare analysis, Currarini et al. (2009) invoke particular forms for the agents' utilities. Under such forms, they obtain that, provided that (i) same-type and different-type links are substitutes, and (ii) the marginal benefits of same-type links are the highest possible, a pattern of complete segregation maximizes welfare. This insight is clearly in consonance with our result that, for high enough assortative interests, maximally homophilic networks are efficient. Secondly, Baccara and Yariv (2013) consider a setting in which an agent's type captures her inclination towards either of two public projects. Then, in stable situations, agents that are exogenously similar end up endogenously in a common group. In this respect, their model delivers a certain degree of (endogenous) homophily. In addition, for an application of theirs, in which connections allow for information sharing, fully segregated groups composed by agents of the same type can emerge only when types are sufficiently different. Although our modeling choice is very different, such an implication is in consonance with our result that maximally homophilic networks arise as stable only if interests for making friends lean strongly towards assortativity—i.e., $\beta \in (0, \beta_L]$. The analysis of Baccara and Yariv (2013) also obtains that stable groups may be heterogeneous with the particular form that such patterns must not contain only one type of individual. In this vein, our result that stable maximally homophilic networks are characterized by a certain degree of quality of heterophilic connections is also in consonance.

We close this section by commenting on plausible ways of meaningfully extending our model. An interesting modification to our analysis would require us to allow for heterogeneity in the strength of the assortative interests of the agents. This would have strong implications for the emergence of stable structures. In particular, the resulting new setting would be incapable of deriving stable structures in which all individuals behave as in solution [b] in Fig. 1. Stable partially homophilic networks would instead require that some individuals behave as in solutions [c] or [a] in Fig. 1—due to that condition $\beta s_i = d_i$ would no longer be required for all agents for a common β. Interestingly, even under heterogeneous assortative interests, our results in Proposition 1 and Proposition 2 would continue to hold with minor modifications. In particular, consider, without loss of generality, that $\beta_1 > \beta_2 > \cdots > \beta_n$. Then, the result in Proposition 1 would continue to hold if $\beta_1 \leq \hat{\beta}(x)$—i.e., if the individual with the lowest assortative interests values relatively more same-type links than different type ones. Similarly, the result of Proposition 2 would continue to hold if $\beta_n \geq \tilde{\beta}(x)$—i.e., if the individual with the highest assortative interests values relatively more different-type links than same-type ones.

Another plausible modification pertains the linear technology for linkage quality assumed in Eq. (1). One could envision the presence of complementaries between individuals of different characteristics as being aptly captured by a technology that explicitly features complementarities between the mutual investments x_{ij} and x_{ji} made by agents i and j of different characteristics. Considering complementarities in the linkage quality technology, however, would affect drastically the tractability of the analysis in the current setting. Notice that our linear technology assumption enables us to work with a "manageable" mapping from the set of possible profiles X to the (s_i, d_i) space of aggregate link qualities for each agent i. Recall that our general class of preferences u are precisely defined over the (s_i, d_i) space. Under a non-linear technology involving complementarities, the maximization problem described in Eq. (9) of Appendix A and Fig. 1 will become highly intractable. Crucially, adding complementaries to the assumed linkage quality production (when individuals belongs to different groups) would naturally enhance heterophilic behavior in stable patterns. We already obtain that type of qualitative insights using instead a simple linear technology with no such complementarities. In this regard, our proposal seeks precisely to avoid this sort of (technology-driven) counter-effects to enhance heterophilic behaviors when assortative interests prevail. In a way, our setting essentially captures the presence of complementaries by embedding them directly in our preference specification. High values of β give us precisely a preference for complementary investments even though such complementary investments do not enhance the "physical" quality of the link.

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Appendix A

The Individual Maximization Problem.—

$$\left.\begin{array}{ll}\max\limits_{\{s_i,d_i\}} & u(s_i,d_i) \\ \text{s.t.:} & I_i^s \le s_i \le (n_\theta - 1)/2 + I_i^s; \\ & I_i^d \le d_i \le n_{\theta'}/2 + I_i^d; \\ & s_i + d_i \le R/2 + I_i^s + I_i^d \end{array}\right\} \quad D_i(x_{-i}), \tag{9}$$

where

$$D_i(x_{-i}) \equiv \big\{(s_i,d_i) \in [I_i^s(x_{-i}), (n_\theta - 1)/2 + I_i^s(x_{-i})] \times [I_i^d(x_{-i}), n_{\theta'}/2 + I_i^d(x_{-i})] \\ \text{s.t.:} \quad s_i + d_i \le R/2 + I_i^s(x_{-i}) + I_i^d(x_{-i})\big\}$$

gives us the set of same-type and different-type qualities (s_i,d_i) feasible for agent i, given a profile of investment strategies x_{-i}.

Let $\phi_i : X_{-i} \to X_i$ denote the *best-response of agent i*, specified as

$$\phi_i(x_{-i}) \equiv \{x_i \in X_i \mid u(s_i(x_i,x_{-i}), d_i(x_i,x_{-i})) \ge u(s_i(x_i',x_{-i}), d_i(x_i',x_{-i})) \quad \forall x_i' \in X_i\}.$$ [28]

Accordingly, let $\Phi : X \to X$, where $\Phi = (\phi_1,\dots,\phi_n)$, be the *best-response correspondence of all agents* in the society. Then, the Nash stability condition, imposed by 1. of Definition 1 above, can be equivalently expressed as requiring that x^* that satisfies the classical fixed point condition $x^* \in \Phi(x^*)$.

Since u is smooth and concave, and $X \subset \mathbb{R}^{n(n-1)}$ is a compact real set, the best-response correspondence Φ of the agents in the population is upper hemi-continuous. Furthermore the correspondence Φ satisfies that $\Phi(x)$ is non-empty, closed, and convex for each profile $x \in X$. By Kakutani's fixed point theorem, it then holds that, under the capacity constraints in Assumption 3, a Nash stable network always exists for any preference specification u that satisfies Assumption 2 in our network formation game Γ.

Appendix B

Omitted Proofs.—

Proof of *Lemma 2*. Consider a strategy profile x that induces a Nash stable friendship network $g = g(x)$. For an arbitrary agent i, let $(s_i(x), d_i(x))$ be the aggregate same and

[28] Or equivalently, $\phi_i(x_{-i}) \equiv \{x_i \in X_i \mid \pi_i(g(x_i,x_{-i})) \ge \pi_i(g(x_i',x_{-i})) \quad \forall x_i' \in X_i\}$.

different-type qualities induced by x. By Assumption 3, for agent $i \in N_\theta$, $R - n_{\theta'} \geq 1$ if i invests with full intensity in all others of different type and $R - (n_\theta - 1) > 1$ if i invests with full intensity in all others of her same type. Since $n_\theta \geq 3$ for each type $\theta \in \Theta$, we can consider a deviation by a pair of agents $i, j \in N$ of different type, θ and θ', respectively, described as follows:

(i) Consider that neither agent i nor agent j invest one unit into each other, that is, $x_{ij} \in [0,1)$ and $x_{ji} \in [0,1)$. Consider that agent i decreases the sum of her investments in agents from $N_{\theta'} \setminus \{j\}$ by an amount $\varepsilon_i > 0$ and, at the same time, agent j decreases the sum of her investments in agents from $N_\theta \setminus \{i\}$ by an amount $\varepsilon_j > 0$. Notice that agent i can always decrease invesments in this way as $R - (n_\theta - 1) > 1$ or, alternatively, $\sum_{k \in N_{\theta'} \setminus \{j\}} x_{ik} > 0$. The argument is analogous for agent j.

(ii) Agent i invests the saved amount ε_i in agent j and, at the same time, agent j invests the saved amount ε_j in agent i. With this class of deviations we obtain new values $d'_i = d_i + \varepsilon_j$ and $d'_j = d_j + \varepsilon_i$ for the total qualities of different-type links, while the total qualities of same-type links s_i and s_j remain unchanged.

By the monotonicity of preferences, i and j strictly benefit form this class of joint deviations. Such deviations are avoided if the strategy profile x does not allow for the re-invesments decribed in (ii). Notice that the only way to avoid such re-invesments is to require that for each pair of different agents, at least one of the agents is already investing with full intensity in the other agent.

We now show that if the aforementioned class of deviations is avoided, there is no other joint deviation from profile x such that both agents from a given pair strictly benefit. In particular, any other type of deviation leaves at least one of the agents indifferent or worse off than before the deviation. Specifically, consider a pair of agents i and j of different type, θ and θ', respectively:

1. Let i and j be such that $x_{ji} = 1$ and $x_{ij} \in [0,1)$. Notice that as x_{ij} is part of $i's$ (unilateral) optimal strategy, it must be that either: (i) $\beta = d_i(x)/s_i(x)$, that is, i behaves as in [b] of Fig. 1 or (ii) $\beta < d_i(x)/s_i(x)$, that is, i behaves as in [a] of Fig. 1. Consider that i and j jointly deviate. Denote the new strategies of i and j, by x'_i and x'_j, respectively and the new strategy profile by x'. Consider that the new strategy of agent j in particular satisfies, $x'_{ji} = x_{ji}$ with $x' \neq x$. That is, j does not alter her invesment in i. Notice that for i, given her new strategy x'_i, there are three possible scenarios: (a) if x'_i is such that she does not change her invesment in same-type and different type others, she is as well off as before the deviation, (b) if x'_i is such that i increases her invesment in different-type others and hence decreases her invesment in same-type others, it then follows that $d_i(x') > d_i(x)$ and $s_i(x') < s_i(x)$. Then there are two options according to the above cases: (i) $\beta = d_i(x)/s_i(x) < d_i(x')/s_i(x')$ and thus same-type relationships become marginally more valuable than different-type ones or *(ii)* $\beta < d_i(x)/s_i(x) < d_i(x')/s_i(x')$ and thus same-type relationships are marginally more valuable than different-type ones.

Then, in either case i is worse off than before the deviation. Finally, (c) x'_i is such that i reduces her invesment in different-type others and increases her invesment in same-type others. Then, in the above scenario (i) $d_i(x')/s_i(x') < \beta = d_i(x)/s_i(x)$ and thus different-type relationships are marginally more valuable than same-type ones. Thusagain, i is worse off than before the deviation.[29]

2. Let i and j be such that $x_{ij} = 1$ and $x_{ji} = 1$. Notice that in this case i may be behaving unilaterally as in: (i) [b], (ii) [a] or (iii) [c] of Fig. 1. Consider that i and j deviate to the new strategies x'_i and x'_j. For agent j, let $x'_{ji} = x_{ji}$ with $x' \neq x$ as in case 1 above. There are three possible scenarios: (a) if x'_i is such that she does not change her invesment in same-type and different type others, she is as well off as before the deviation, (b) if x'_i is such that i increases her invesment in different-type others and hence decreases her invesment in same-type others, then there are two options according to the scenarios (i) and (ii) above, and the reasoning is exactly analogous as in case 1. (b), thus, i is worse off than before the deviation. Finally, (c) x'_i is such that i reduces her invesment in different type others and increases her invesment in same-type others. It then follows that $d_i(x') < d_i(x)$ and $s_i(x') > s_i(x)$. Then there are two options according to the above scenarios: (iii) $d_i(x')/s_i(x') < d_i(x)/s_i(x) < \beta$ or (i) $d_i(x')/s_i(x') < \beta = d_i(x)/s_i(x)$. In both scenarios different-type relationships are marginally more valuable than same-type ones. Thus, i is worse off than before the deviation.

The case in which j reduces her invesment in i, so that x'_j particularly entails that $x'_{ji} = x_{ji} - \varepsilon_j$, $\varepsilon_j > 0$, is analogous. Moreover, in (a) of cases 1 and 2 above, agent i becomes even worse off than before the deviation as she loses different-type investments.

The case in which i and j are of the same type θ is also analogous. We therefore omit further details.

∎

PROOF OF *Proposition 1*. Consider a strategy profile x that induces a maximally homophilic network $g = g(x)$. Hence, for each agent $i \in N$ of type θ, we have $x_{ij} = 1$ for each $j \in N^i_\theta$ and $\sum_{j \in N_{\theta'}} x_{ij} = R - (n_\theta - 1)$ for the type $\theta' \neq \theta$.

1. *Robustness against unilateral deviations:* First, it directly follows that $I^s_i(x_{-i}) = (1/2)(n_\theta - 1)$ for each agent i of type θ. Then, the particular value $\underline{\beta}(\theta; x_{-i})$ of the slope β specified in Eq. (3), under which each agent i of type θ is indifferent between investing with full intensity in links to each other agent of her same type and investing less, equals:

$$\underline{\beta}(\theta; x_{-i}) = \frac{R - (n_\theta - 1) + 2I^d_i(x_{-i})}{2(n_\theta - 1)}.$$

[29] Notice that if i already behaves as in [a] of Fig. 1, the scenario (ii) above, where $\beta < d_i(x)/s_i(x)$ and i is investing with full intensity in same type others, case (c) cannot take place.

Therefore, if for each possible type $\theta \in \Theta$, and each type $\theta' \neq \theta$, the level β of assortative interests equals the indifference cutoff value $\underline{\beta}(\theta; x_{-i})$ above, then no agent has unilateral incentives to deviate from the proposed strategy profile x, as stated by condition 1. of the proposition. On the other hand, if $\beta > \underline{\beta}(\theta; x_{-i})$, then such an agent $i \in N_\theta$ has incentives to deviate from investing with full intensity in each other agent of her same type. Thus, the inequality $\beta \leq \underline{\beta}(\theta; x_{-i})$, for each $i \in N_\theta$ and each type $\theta \in \Theta$, gives us a necessary condition for x to be stable.

2. *Robustness against bilateral deviations:* Note first that if $\beta \leq \underline{\beta}(\theta; x_{-i})$, then no agent of type θ has incentives to lower her full-intensity investments in each other agent of her same type. Therefore, no pair of two different agents of the same type have incentives to deviate from investing with full intensity in each other agent of type θ either. The only possible class of profitable bilateral deviations that remains to be ruled out must then involve two agents of different types. In particular, since $n_\theta \geq 3$ for each type θ, we can consider a deviation by a pair of agents i and j, with $i \in N_A$ and $j \in N_B$, in which each of the two agents redirect third-party investments into each other. As already argued in the proof of Lemma 2, such a (unique) class of bilateral deviations is prevented if and only if for each pair of agents that belong to different groups at least one of the agents invests with full intensity in the other agent, as stated in 2. of the proposition.

Finally, we verify that the size of the resource R allows for the type of connections described in conditions 1. and 2. of the proposition to be feasible for all agents in the population. Note that under a strategy profile x that induces a maximally homophilic network, the capacity constraint requirement (Assumption 3) for each agent $i \in N_\theta$, for $\theta \in \Theta$, takes the form

$$(n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ij} \leq R.$$

By aggregating the requirement above across all agents $i \in N_\theta$, for both types $\theta \in \Theta$, it follows then that the size of the resource R must necessarily satisfy

$$n_A(n_A - 1) + n_B(n_B - 1) + \sum_{i \in N_A} \sum_{j \in N_B} x_{ij} + \sum_{i \in N_B} \sum_{j \in N_A} x_{ij} \leq nR. \tag{10}$$

Note that the number of possible pairs $(i, j) \in N_A \times N_B$ of different-type agents is $n_A n_B$. If for each of such $n_A n_B$ different possible pairs at least one of the agents from the pair invests with full intensity in the other agent, as prescribed by condition 2. of the proposition, then the minimum aggregate quality for the connections among different-type agents amounts precisely to $n_A n_B$. Therefore, any profile x that satisfies such a condition must necessarily satisfy $n_A n_B \leq \sum_{i \in N_A} \sum_{j \in N_B} x_{ij} + \sum_{i \in N_B} \sum_{j \in N_A} x_{ij}$. By combining this inequality with the condition in Eq. (10) above, we obtain that a necessary requirement from condition 2 of the proposition to be satisfied is $n_A(n_A - 1) + n_B(n_B - 1) + n_A n_B \leq nR$, or equivalently, $R \geq (n - 1) - n_A n_B / n$. ∎

PROOF OF *Corollary 2*. The sufficient conditions for β and R derived by Corollary 2 follow from the requirements of Proposition 1. First, note that for the class of strategy profiles x proposed in the corollary, we have $I_i^d(x_{-i}) \geq (n_B - 1)/2$ for each agent $i \in N_\theta$ and each type $\theta \in \Theta$. Then, by combining the lower bound $(n_B - 1)/2$ on the total incoming intensity $I_i^d(x_{-i})$ with the condition 1. derived in Proposition 1, it follows that $\beta \leq [R + (n_B - n_A)]/2(n_A - 1)$ is a sufficient condition for all agents to have incentives to invest with full intensity in each other same-type agent. Secondly, note that condition 2. of Proposition 1 is satisfied by construction for the strategy profiles x described by the corollary. Thirdly, consider an agent $i \in N_\theta$, for $\theta \in \Theta$, who makes investments as prescribed by the class of strategy profiles x proposed in Corollary 2 but does not invest any extra amount on any other different-type agent. Then, it follows that x_i satisfies

$$\sum_{j \in N_\theta^i} x_{ij} + \sum_{j \in N_{\theta'}} x_{ij} = (n_\theta - 1) + n_{\theta'}/2 \quad \text{for } n_{\theta'} \text{ even;}$$
$$\sum_{j \in N_\theta^i} x_{ij} + \sum_{j \in N_{\theta'}} x_{ij} = (n_\theta - 1) + (n_{\theta'} - 1)/2 + 1 \quad \text{for } n_{\theta'} \text{ odd.}$$

In addition, we know that for each agent $i \in N_\theta$ and each type $\theta \in \Theta$

$$(n_\theta - 1) + (n_{\theta'} - 1)/2 + 1 \leq n_A + (n_B - 1)/2.$$

Therefore, if $R \geq n_A + (n_B - 1)/2$, then each agent has the amount of resource R required to follow the prescription for the class of strategy profiles x proposed by Corollary 2. ∎

PROOF OF *Proposition 2*. Consider a strategy profile x that induces a minimally homophilic network $g = g(x)$. Hence, for each agent $i \in N$ of type θ, we have $x_{ij} = 1$ for each $j \in N_{\theta'}$ and $\sum_{j \in N_\theta^i} x_{ij} = R - n_{\theta'}$ for the type $\theta' \neq \theta$.

1. *Robustness against unilateral deviations:* It follows directly that $I_i^d(x_{-i}) = (1/2)n_{\theta'}$ for each agent i of type θ. Then, the particular value $\overline{\beta}(\theta; x_{-i})$ of the slope β specified in Eq. (3), under which each agent i of type θ is indifferent between investing with full intensity in links to each different-type agent and investing less, equals:

$$\overline{\beta}(\theta; x_{-i}) = \frac{2n_{\theta'}}{(R - n_{\theta'}) + 2I_i^s(x_{-i})}.$$

Therefore, if for each possible type $\theta \in \Theta$, and each type $\theta' \neq \theta$, the level β of assortative interests equals the indifference value $\overline{\beta}(\theta; x_{-i})$ above, then no agent has unilateral incentives to deviate from the proposed strategy profile x, as stated by condition 1. of the proposition. On the other hand, if $\beta < \overline{\beta}(\theta; x_{-i})$, then such an agent $i \in N_\theta$ has incentives to deviate from investing with full intensity in each different-type agent. Thus, the inequality $\beta \geq \overline{\beta}(\theta; x_{-i})$, for each $i \in N_\theta$ and each type $\theta \in \Theta$, gives us a necessary condition for x to be stable.

2. *Robustness against bilateral deviations:* Note first that if $\beta \geq \overline{\beta}(\theta; x_{-i})$, then no agent of type θ has incentives to lower her full-intensity investments in each agent of type θ'. Therefore, no pair of two different agents have incentives to deviate from investing with full intensity in each other either. The only possible class of profitable bilateral deviations that remains to be ruled out must then involve two agents of the same type. In particular, since $n_\theta \geq 3$ for each type $\theta \in \Theta$, we can consider a deviation by a pair of agents $i, j \in N_\theta$, for $i \neq j$, in which both agents redirect third-party investments into each other. By the proof of Lemma 2, such a (unique) class of bilateral deviations is prevented if and only if for each pair of same-type agents, at least one of them is already investing with full intensity in the other agent.

Finally, the size of the resource R must allow for the type of connections described in conditions 1. and 2. of the proposition to be feasible for all agents in the population. Under a strategy profile x that induces a minimally homophilic network, the capacity constraint requirement (Assumption 3) for each agent $i \in N_\theta$, for $\theta \in \Theta$, takes the form

$$\sum_{j \in N_\theta \setminus \{j\}} x_{ij} + n_{\theta'} \leq R.$$

By aggregating the requirement above across all agents $i \in N_\theta$, for both types $\theta \in \Theta$, it follows then that the size of the resource R must necessarily satisfy

$$\sum_{i \in N_\theta} \sum_{j \in N_\theta \setminus \{j\}} x_{ij} + n_\theta n_{\theta'} \leq n_\theta R. \tag{11}$$

Note that the number of possible pairs $(i, j) \in N_\theta \times N_\theta$, with $i \neq j$, between same-type agents is $n_\theta(n_\theta - 1)$. If for each of such $n_\theta(n_\theta - 1)$ different possible pairs, at least one of the agents from the pair invests with full intensity in the other agent, as prescribed by condition 2. in the proposition, it follows that the aggregate quality between all the agents of type θ must be at least $n_\theta(n_\theta - 1)/2$. Therefore, a minimally homophilic network that satisfies such a condition must necessarily satisfy $n_\theta(n_\theta - 1)/2 \leq \sum_{i \in N_\theta} \sum_{j \in N_\theta \setminus \{j\}} x_{ij}$. By combining this inequality with the condition in Eq. (11) above, we obtain that $(n_\theta - 1)/2 + n_{\theta'} \leq R$ for each $\theta \in \Theta$ is a necessary requirement for condition 2. of the proposition to be satisfied. Since $n_A \geq n_B$, it follows that $R \geq n_A + (n_B - 1)/2$ must in fact hold. ∎

PROOF OF *Corollary 4*. The sufficient conditions for β and R derived by Corollary 4 follow from the requirements of Proposition 2. First, note that, for the class of strategy profiles x proposed in the corollary, we have $I_i^s(x_{-i})$ is always lower for agents $i \in N_B$ than for agents $i \in N_A$. Also, for each agent $i \in N_B$, we have $I_i^s(x_{-i}) = R - n_A$. Then, by combining the total incoming intensity $I_i^s(x_{-i})$ with the condition 1. derived in Proposition 2, it follows that $\beta \geq n_A/(R - n_A)$ is a sufficient condition for all agents to have incentives to invest with full intensity in each different-type agent. Secondly, it is easy to verify that, by construction, the proposed strategy profile always satisfies the key condition given in

Lemma 2 to prevent profitable bilateral deviations. Finally, since $n_A \geq n_B$, it follows that $l_A = n_A - n_B$ for each $i \in N_A$, whereas $l_B = 0$ for each $i \in N_B$ Therefore, if $R \geq n_A + \alpha(n_B)$, then we can ensure that each agent has, at least, the amount R of the resource required to follow the prescription for the class of strategy profiles x proposed by Corollary 4. ▮

PROOF OF PROPOSITION 3. We prove statements (i) and (ii) of the proposition by contradiction.

(i) Consider a strategy profile x that induces a maximally homophilic network $g = g(x)$. Then, for each agent $i \in N_\theta$, and each type $\theta \in \Theta$, we have $x_{ij} = 1$ for each $j \in N_\theta^i$. Therefore, $I_i^s(x_{-i}) = (1/2)(n_\theta - 1)$ for each agent $i \in N_\theta$, and each type $\theta \in \Theta$. Then, using the expression of the upper bound $\underline{\beta}(\theta; x_{-i})$ for the indifference value of β, associated to the unilateral optimal choice described by [a] in Fig. 1, it follows that

$$\hat{\beta}_{i,\theta}(x) \equiv \frac{R - (n_\theta - 1) + 2I_i^d(x_{-i})}{2(n_\theta - 1)}. \tag{12}$$

That is the value for the level of assortative interests under which agent i is indifferent between investing with full intensity in each other same-type agent and investing lower amounts in some same-type agent. First, suppose that the strategy profile x is such that $x_{ji} = [R - (n_{\theta'} - 1)]/n_\theta$ for each pair of agents $i \in N_\theta$, and $j \in N_{\theta'}$, for each type $\theta \in \Theta$ and the type $\theta' \neq \theta$. Thus, each agent in the population receives a constant proportional amount of investments from each different-type agent. In this case the investment received by each agent from each different-type agent depends only on the group to which she belongs. For each $i \in N_\theta$ and each $\theta \in \Theta$, the indifference value in Eq. (12) takes the form

$$\hat{\beta}_\theta \equiv \frac{nR - n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1)}{2(n_\theta - 1)n_\theta}.$$

Suppose that $\beta \in (1, +\infty)$. Then, each agent $i \in N_\theta$ has (weak) incentives to invest with full intensity in each other same-type agent only if

$$\hat{\beta}_\theta > 1 \;\Leftrightarrow\; nR - 3n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1) > 0.$$

Now, recall that by Assumption 3, $R < n_\theta + n_{\theta'} - 1$. Therefore, we know that

$$\begin{aligned} nR - 3n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1) &< (n_\theta + n_{\theta'})(n_\theta + n_{\theta'} - 1) - 3n_\theta(n_\theta - 1) - n_{\theta'}(n_{\theta'} - 1) \\ &= 2(n'_\theta - n_\theta + 1) < 0 \text{ for some type } \theta \in \Theta, \text{ and for } \theta' \neq \theta, \end{aligned}$$

since $n_A \geq n_B$. Therefore, each agent $i \in N_A$ has (strict) incentives to deviate from the proposed profile x that induces a maximally homophilic network. Secondly, consider another strategy profile $x' \neq x$ that induces as well a maximally homophilic network $g = g(x')$ and such that $\tilde{\beta}_{i,\theta}(x') > \tilde{\beta}_\theta$. By the monotonicity of preferences, the resource constraint $\sum_{j \neq i} x'_{ij} \leq R$ must be satisfied with equality for each agent who has no unilateral incentives to deviate from x'. Then, it must be the case that $\tilde{\beta}_{j,\theta}(x') < \tilde{\beta}_\theta$ for some other agent

$j \in N_\theta$. In other words, if $\tilde{\beta}_{i,\theta}(x') > 1$ for some agent $i \in N_\theta$, then it must be the case that $\tilde{\beta}_{j,\theta}(x') < 1$ for some other agent $j \in N_\theta$. In this case, such an agent j would have (strict) incentives to deviate unilaterally from x'. Therefore, we conclude that if $\beta \in (1, +\infty)$, then at least one agent in the population has unilateral incentives to deviate from any profile that induces a maximally homophilic network.

(ii) Consider a strategy profile x that induces a minimally homophilic network $g = g(x)$. Then, for each agent $i \in N_\theta$, and each type $\theta \in \Theta$, we have $x_{ij} = 1$ for each $j \in N_{\theta'}$ for the type $\theta \neq \theta$. Therefore, $I_i^d(x_{-i}) = (1/2)n_{\theta'}$ for each agent $i \in N_\theta$, and each type $\theta \in \Theta$. Then, using the expression of the upper bound $\overline{\beta}(\theta; x_{-i})$ for the indifference value of β, associated to the unilateral optimal choice described by [c] in Fig. 1, it follows that

$$\tilde{\beta}_{i,\theta}(x) \equiv \frac{2n_{\theta'}}{(R - n_{\theta'}) + 2I_i^s(x_{-i})}. \tag{13}$$

That is the value for the level of assortative interests under which agent i is indifferent between investing with full intensity in each different-type agent and investing lower amounts in some different-type agent. First, suppose that the strategy profile x is such that $x_{ji} = (R - n_{\theta'})/(n_\theta - 1)$ for each pair $i, j \in N_\theta$, with $i \neq j$, and for each type $\theta \in \Theta$. Thus, each agent in the population receives a constant proportional amount of investments from each other same-type agent. In this case, the investment received by each agent from each other same-type agent depends only on the group to which she belongs. For each $i \in N_\theta$ and each $\theta \in \Theta$, the indifference value in Eq. (13) takes the form

$$\tilde{\beta}_\theta \equiv \frac{n_{\theta'}}{(R - n_{\theta'})}.$$

Suppose that $\beta \in (0, 1]$. Then, each agent $i \in N_\theta$ has (weak) incentives to invest with full intensity in each different-type agent only if $\tilde{\beta}_\theta \leq 1$. Using the expression for $\tilde{\beta}_\theta$ derived above, we observe that this is possible for each type $\theta \in \Theta$ only if $R > 2n_A$ and $R > 2n_B$ simultaneously. However, that is a contradiction given that by Assumption 3, $R < n - 1$. Secondly, consider another strategy profile $x' \neq x$ that induces as well a minimally homophilic network $g = g(x')$ and such that $\tilde{\beta}_{i,\theta}(x') < \tilde{\beta}_\theta$. By the monotonicity of preferences, the resource constraint $\sum_{j \neq i} x'_{ij} \leq R$ must be satisfied with equality for each agent who has no unilateral incentives to deviate from x'. Then, it must be the case that $\tilde{\beta}_{j,\theta}(x') > \tilde{\beta}_\theta$ for some other agent $j \in N_\theta$. In other words, if $\tilde{\beta}_{i,\theta}(x') \leq 1$ for some agent $i \in N_\theta$, then it must be the case that $\tilde{\beta}_{j,\theta}(x') > 1$ for some other agent $j \in N_\theta$. In this case, such an agent j would have strict incentives to deviate unilaterally from x'. Therefore, we conclude that if $\beta \in (0, 1]$, then at least one agent in the population has unilateral incentives to deviate from any profile that induces a minimally homophilic network. ∎

PROOF OF *Corollary 5*. First, consider a strategy profile x that induces a maximally homophilic network $g = g(x)$. Stability of such a network requires that no agent wants to deviate unilaterally from the proposed strategy profile x. Specifically, stability of a maximally homophilic network $g = g(x)$ requires that

$$\beta \leq \hat{\beta}(x) = \inf_{i \in N_\theta, \theta \in \Theta} \frac{R - (n_\theta - 1) + \sum_{j \in N_{\theta'}} x_{ji}}{2(n_\theta - 1)}.$$

Now, let us complete the description of the proposed profile x by requiring that each agent $i \in N_\theta$, for each $\theta \in \Theta$, receives a common intensity of investments from the different-type agents. Thus, consider that x satisfies $\sum_{j \in N_{\theta'}} x_{ji} = n_{\theta'}[R - (n_{\theta'} - 1)]/n_\theta$ for each $i \in N_\theta$ and each $\theta \in \Theta$. This construction of x entails that the highest possible cutoff value $\hat{\beta}(x)$ for the optimal unilateral behavior where agents want to invest with full intensity in all other same-type agents (which was described by [a] of Fig. 1) cannot exceed one. Then, given that $n_A \geq n_B$, we observe that such a proposed maximally homophilic network $g = g(x)$ satisfies the criterion of robustness against unilateral deviations if and only if

$$\beta \leq \frac{nR - [n_A(n_A - 1)) + n_B(n_B - 1)]}{2n_A(n_A - 1)} \equiv \beta_l.$$

By the proof of Proposition 3 (i), the cutoff value β_l in the right-hand side of the expression above cannot exceed one. Furthermore, Proposition 3 established that if $\beta \leq 1$, then stable minimally homophilic networks do not exist. Then, provided that the cutoff value β_l is strictly less than one, if $\beta \in (\beta_l, 1]$, all stable networks must necessarily be partially homophilic. For the case in which β_l equals one, recall that Proposition 3 guaranteed then that stable maximally homophilic networks do not exist for $\beta > 1$.

Secondly, consider a strategy profile x that induces a minimally homophilic network $g = g(x)$. Recall that robustness against unilateral deviations requires that

$$\beta \geq \tilde{\beta}(x) = \sup_{i \in N_\theta, \theta \in \Theta} \frac{2n_{\theta'}}{(R - n_{\theta'}) + \sum_{j \in N_i^\theta} x_{ji}}.$$

Let us complete the description of the proposed profile x by requiring that each agent $i \in N_\theta$, for each $\theta \in \Theta$, receives a common intensity of investments from the same-type agents. Thus, consider that x satisfies $\sum_{j \in N_i^\theta} x_{ji} = R - n_{\theta'}$ for each $i \in N_\theta$ and each $\theta \in \Theta$. This proposal gives us a profile x that yields the lowest possible cutoff value $\tilde{\beta}(x)$ for the optimal unilateral behavior where all agents want to invest with full intensity in all different-type agents (which was described by [c] of Fig. 1). Since $n_A \geq n_B$, it follows that such a proposed minimally homophilic network $g = g(x)$ satisfies the criterion of robustness against unilateral deviations if and only if

$$\beta \geq \frac{n_A}{R - n_A} \equiv \beta_h > 1.$$

It follows from Proposition 3 (ii), that if $\beta > 1$, then stable maximally homophilic networks do not exist. As a consequence, we know that if $\beta \in (1, \beta_h]$, then all stable networks must necessarily be partially homophilic .

This completes our derivation of an interval $[\beta_l, \beta_h]$ of "intermediate" assortative levels for which only partially homophilic networks are stable. ∎

PROOF OF *Proposition 4*. Consider the social value function v, defined in Eq. (7). First, consider an arbitrary investment profile $x \in X$ that induces a collection of sets of pairs $(\{(s_i(x), d_i(x))\}_{i \in N_A}, \{(s_j(x), d_j(x))\}_{j \in N_B})$ of same-type and different-type qualities. Then, the average of the qualities for same-type and different-type links, respectively, across all agents in each group N_θ can be computed as

$$\bar{s}_\theta(x) = (1/n_\theta) \sum_{i \in N_\theta} s_i(x) = (1/2n_\theta) \sum_{i \in N_\theta} \sum_{j \in N_\theta^i} [x_{ij} + x_{ji}] \tag{14}$$

and

$$\bar{d}_\theta(x) = (1/n_\theta) \sum_{i \in N_\theta} d_i(x) = (1/2n_\theta) \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} [x_{ij} + x_{ji}]. \tag{15}$$

Secondly, using the definition of same-type $s_i(x)$ and different-type $d_i(x)$ aggregate qualities, let us propose another investment profile $\hat{x} \in X$ such that $s_i(\hat{x})$ and $d_i(\hat{x})$ be constant across all agents $i \in N_\theta$ for each type $\theta \in \Theta$. From the definition of the aggregate qualities, it follows that the quantities $\sum_{j \in N_\theta^i} \hat{x}_{ij}$, $\sum_{j \in N_\theta^i} \hat{x}_{ji}$, $\sum_{j \in N_{\theta'}} \hat{x}_{ij}$, and $\sum_{j \in N_{\theta'}} \hat{x}_{ji}$ must be constant across agents within each population group. Accordingly, we start by proposing a profile $\hat{x}$ such that, for each agent $i \in N_\theta$ and each type $\theta \in \Theta$, we have

(a) $\sum_{j \in N_\theta^i} \hat{x}_{ij} = y_{\theta\theta}$ and $\sum_{j \in N_\theta^i} \hat{x}_{ji} = z_{\theta\theta}$, and

(b) $\sum_{j \in N_{\theta'}} \hat{x}_{ij} = y_{\theta\theta'}$ and $\sum_{j \in N_{\theta'}} \hat{x}_{ji} = z_{\theta\theta'}$.

In particular, for any agent $i \in N_\theta$, the amount $y_{\theta\theta}$ describes i's total investments in the rest of her same-type agents, whereas $y_{\theta\theta'}$ describes i's aggregate investments in all different-type agents. Similarly, for any agent $i \in N_\theta$, the amount $z_{\theta\theta}$ describes the total of investments that i receives from the rest of her same-type agents, whereas $z_{\theta\theta'}$ describes i's aggregate investments that i receives from all different-type agents. Thus, under the profile $\hat{x}$, the sums of the aggregate outgoing and incoming investments are only contingent on the characteristics of the agents.

Given the proposal above, note first that, by summing the investments made and received over all same-type agents for any type, it follows that $n_\theta y_{\theta\theta} = n_\theta z_{\theta\theta}$, so that it must necessarily be the case that $y_{\theta\theta} = z_{\theta\theta}$. Secondly, by noting that the sum of the investments made by all agents from N_θ in all the agents of the group $N_{\theta'}$ must be equal to the sum of the investments received by all agents of the group $N_{\theta'}$ from all agents from N_θ, it follows $n_\theta y_{\theta\theta'} = n_{\theta'} z_{\theta'\theta}$. Our proposal accordingly incorporates also these two considerations. Crucially, from the definitions of s_i and d_i, it follows that the characteristics imposed by our proposal for the profile $\hat{x}$ are necessary and sufficient to make

$s_i(\hat{x}) = s_j(\hat{x})$ and $d_i(\hat{x}) = d_j(\hat{x})$ for each pair of (distinct) agents $i, j \in N_\theta$, for each type $\theta \in \Theta$.

Furthermore, consider that, for such a profile $\hat{x}$, each agent satisfies her capacity constraint (Assumption 3) with equality. Then, the constant investments proposed by means of $\hat{x}$ must satisfy

$$y_{\theta\theta} + y_{\theta\theta'} = R \quad \text{for each } \theta \in \Theta, \quad \text{and for } \theta' \neq \theta. \tag{16}$$

The associated qualities are simply derived as $s_i(\hat{x}) = (1/2)[y_{\theta\theta} + z_{\theta\theta}] = y_{\theta\theta}$ and $d_i(\hat{x}) = (1/2)[y_{\theta\theta'} + z_{\theta\theta'}]$, where, as indicated above, we must also consider that $z_{\theta\theta'} = (n_{\theta'}/n_\theta)y_{\theta'\theta}$, for each agent $i \in N_\theta$, each type $\theta \in \Theta$, and $\theta' \neq \theta$.

Now, we can set a relationship between the linkage qualities associated to $\hat{x}$, which are constant across all agents within each population group, and the average qualities derived in Eq. (14) and Eq. (15) for the profile x. By requiring $s_i(\hat{x}) = \bar{s}_\theta(x)$ and $d_i(\hat{x}) = \bar{d}_\theta(x)$ for each $i \in N_\theta$ and each $\theta \in \Theta$, we obtain

$$\begin{aligned} y_{\theta\theta} &= \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_\theta^i} x_{ij}, \quad z_{\theta\theta} = \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_\theta^i} x_{ji}, \quad \text{and} \\ y_{\theta\theta'} &= \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} x_{ij}, \quad z_{\theta\theta'} = \frac{1}{n_\theta} \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} x_{ji}. \end{aligned} \tag{17}$$

Conditional on the above established relationship (Eq. (17)) between the profiles x and $\hat{x}$, clearly the profile $\hat{x}$ satisfies the capacity condition required by Eq. (16):

$$y_{\theta\theta} + y_{\theta\theta'} = (1/n_\theta)\Big(\sum_{i \in N_\theta} \sum_{j \in N_\theta^i} x_{ij} + \sum_{i \in N_\theta} \sum_{j \in N_{\theta'}} x_{ij}\Big) = R.$$

Notice also that our relationship between the two investment profiles, entails that $\hat{x}$ satisfies $\hat{x}_{ij} \in [0,1]$ for each pair of (distinct) agents $i, j \in N$.

Therefore, we establish the key equality $u(s_i(\hat{x}), d_i(\hat{x})) = u(\bar{s}_\theta(x), \bar{d}_\theta(x))$ for each agent $i \in N_\theta$, and each $\theta \in \Theta$, where for each i, $(s_i(\hat{x}), d_i(\hat{x})) \in D_i(\hat{x}_{-i})$. Importantly, we can establish such an equality regardless of whether $(\bar{s}_\theta(x), \bar{d}_\theta(x))$ belongs to the feasible set $D_i(x_{-i})$ for each agent $i \in N_\theta$, and each $\theta \in \Theta$. Now, since the utility function u is (strictly) concave in the (s_i, d_i) space (Assumption 2–(3)), it follows that

$$\begin{aligned} v(g(\hat{x})) &= \sum_{i \in N} u(s_i(\hat{x}), d_i(\hat{x})) = \sum_{\theta \in \Theta} \sum_{i \in N_\theta} u(\bar{s}_\theta(x), \bar{d}_\theta(x)) \\ &= \sum_{\theta \in \Theta} \sum_{i \in N_\theta} u\Big((1/n_\theta) \sum_{i \in N_\theta} s_i(x), (1/n_\theta) \sum_{i \in N_\theta} d_i(x)\Big) \\ &\geq \sum_{\theta \in \Theta} \sum_{i \in N_\theta} (1/n_\theta) \sum_{i \in N_\theta} u(s_i(x), d_i(x)) = \sum_{\theta \in \Theta} \sum_{i \in N_\theta} u(s_i(x), d_i(x)) = v(g(x)), \end{aligned}$$

where the inequality above holds strictly unless our initial investment profile x satisfies $s_i(x) = \bar{s}_\theta(x)$ and $d_i(x) = \bar{d}_\theta(x)$ for each $i \in N_\theta$ and each $\theta \in \Theta$. It follows then that an efficient network $g = g(\hat{x})$ requires that the qualities $(s_i(\hat{x}), d_i(\hat{x}))$ be constant across all agents i within each of the two population groups. ∎

PROOF OF *Proposition 5*. Let $\hat{x}$ be a strategy profile that satisfies the necessary condition given by Proposition 4. Then, we can fully describe the profile $\hat{x}$ using the type-contingent aggregate investments y_{AA}, y_{BB}. The social planner can select in a totally independent way the pair of variables y_{AA}, y_{BB}, under the respective restrictions $y_{AA} \in [R - n_B, n_A - 1]$ and $y_{BB} \in [R - n_A, n_B - 1]$. In turn, the aggregated investments y_{AB}, y_{BA}, z_{AB}, and z_{BA} can be derived from the optimally selected quantities y_{AA}, y_{BB}. Using the expression of the social value in Eq. (8), the problem that the social planner can thus be set as

$$\begin{aligned} \max_{\{y_{AA}, y_{BB}\}} \; & n_A u\Big(y_{AA}, \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_A}\Big) + n_B u\Big(y_{BB}, \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_B}\Big) \\ \text{s.t.:} \; & y_{AA} \in [R - n_B, n_A - 1]; \\ & y_{BB} \in [R - n_A, n_B - 1]. \end{aligned} \tag{18}$$

Observation of the problem in Eq. (18) above allows us to proceed as follows.

(i) We identify a sufficient condition on the level of assortative interests β under which, regardless of the aggregate investment choice y_{AA} of the agents from the larger group N_A, the utility of any agent from the smaller group N_B is maximized when she invests with full intensity in all other same-type agents, i.e., $y_{BB} = n_B - 1$. Furthermore, the identified condition on β simultaneously ensures that the agents from the larger group N_A maximize their utilities when they choose to invest with full intensity in all other same-type agents, i.e., $y_{AA} = n_A - 1$, independently of the choice y_{BB} of the agents from the smaller group. Since the welfare function $v(g(\hat{x}))$ aggregates the utilities of all the agents, for each of the two groups, it follows that the derived condition is sufficient to guarantee that the value function is maximized when all agents invest with full intensity in all other same-type agents, i.e., $y_{AA} = n_A - 1$ and $y_{BB} = n_B - 1$.

On the one hand, let us take as given an arbitrary quantity $y_{AA} \in [R - n_B, n_A - 1]$, and suppose then that the social planner chooses the quantity y_{BB} in order to maximize the utility of a representative agent of the smaller group, N_B. Thus, we are now restricting attention to the (hypothetical) problem

$$\max_{y_{BB} \in [R - n_A, n_B - 1]} u\Big(y_{BB}, \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_B}\Big).$$

Then, recall that Assumption 2–(4) (b) establishes that $\partial u(s_i, d_i)/\partial s_i > \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i > \beta$. Therefore, if

$$\beta < \frac{nR - n_A y_{AA} - n_B y_{BB}}{2 n_B y_{BB}}$$

for each $y_{BB} \in [R - n_A, n_B - 1]$, then we can guarantee that maximization of the utility of any agent $i \in N_B$ is uniquely achieved by selecting $y_{BB} = n_B - 1$, for each possible $y_{AA} \in [R - n_B, n_A - 1]$. Furthermore, since the function $[nR - n_A y_{AA} - n_B y_{BB}]/2n_B y_{BB}$ is strictly decreasing in y_{BB}, it follows that

$$\beta < \frac{nR - n_A y_{AA} - n_B(n_B - 1)}{2 n_B (n_B - 1)} \tag{19}$$

is a sufficient condition that ensures maximization of the utility of the agents $j \in N_B$ is characterized by $y_{BB} = n_B - 1$, for any given $y_{AA} \in [R - n_B, n_A - 1]$.

On the other hand, let us now take as given an arbitrary quantity $y_{BB} \in [R - n_A, n_B - 1]$, and restrict attention to the (hypothetical) problem of choosing the value of y_{AA} that solves

$$\max_{y_{AA} \in [R - n_B, n_A - 1]} u\left(y_{AA}, \frac{nR - n_A y_{AA} - n_B y_{BB}}{2 n_A}\right).$$

Using again Assumption 2–(4) (b), we can guarantee the solution to the problem above is characterized by $y_{AA} = n_A - 1$ if

$$\beta < \frac{nR - n_A y_{AA} - n_B y_{BB}}{2 n_A y_{AA}}$$

for each $y_{AA} \in [R - n_B, n_A - 1]$. Since the function $[nR - n_A y_{AA} - n_B y_{BB}]/2n_A y_{AA}$ is strictly decreasing in y_{AA}, it follows that

$$\beta < \frac{nR - n_A(n_A - 1) - n_B y_{BB}}{2 n_A (n_A - 1)} \tag{20}$$

is a sufficient condition that ensures maximization of the utility of the agents $i \in N_A$ is characterized by $y_{AA} = n_A - 1$, for any choice $y_{BB} \in [R - n_A, n_B - 1]$.

Therefore, if both conditions Eq. (19) and Eq. (20) are simultaneously satisfied for values $y_{AA} = n_A - 1$ and $y_{BB} = n_B - 1$, then the utility of the agents from the smaller group N_B is maximized when they choose $y_{BB} = n_B - 1$ conditional on the choice $y_{AA} = n_A - 1$ while, at the same time, the utility of the agents from the larger group N_A is maximized when they choose $y_{AA} = n_A - 1$ conditional on the choice $y_{BB} = n_B - 1$. Since conditions Eq. (19) and Eq. (20) combined guarantee such (common) features for the optimal choices of the two separate (hypothetical) problems—relative to each of the two populations—, we obtain that such sufficient conditions combined ensure that the only solution to the problem in Eq. (18) entails $y_{AA} = n_A - 1$ and $y_{BB} = n_B - 1$.

Since $n_A \geq n_B$, we have that

$$\frac{nR - n_A(n_A-1) - n_B(n_B-1)}{2n_B(n_B-1)} \geq \frac{nR - n_A(n_A-1) - n_B(n_B-1)}{2n_A(n_A-1)}.$$

In addition, recall from Eq. (4) the expression of the particular value

$$\beta_l = [nR - n_A(n_A-1) - n_B(n_B-1)]/2n_A(n_A-1).$$

Thus, if $\beta < \beta_l$, then the only way in which the social planner can maximize the social value $v(g(\hat{x}))$ is by choosing $y_{AA} = n_A - 1$, $y_{BB} = n_B - 1$. Such choices also yield $y_{AB} = R - (n_A - 1)$, $y_{BA} = (R - (n_A - 1)) + (n_A - n_B)$, $z_{AB} = (n_B/n_A)[(R - (n_A - 1)) + (n_A - n_B)]$, and $z_{BA} = (n_A/n_B)(R - (n_A - 1))$. Accordingly, for each agent $i \in N_\theta$, each type $\theta \in \Theta$, and $\theta' \neq \theta$, an efficient network $\hat{g} = g(\hat{x})$ entails

$$s_i(\hat{x}) = n_\theta - 1 \;\text{ and }\; d_i(\hat{x}) = \frac{n(R - (n_A - 1)) + n_B(n_A - n_B)}{2n_\theta}.$$

(ii) Similarly to the arguments used in (i), we consider separately two hypothetical problems that address the maximization of the utility of any agent from a given group, regardless of the choices made by the agents from the other group. Again, we derive a sufficient condition on the level of assortative interests β under which, regardless of the aggregate investment choice y_{AA} of the agents from the larger group N_A, the utility of any agent from the smaller group N_B is maximized when the agents invests with full intensity in all different-type agents, i.e., $y_{BB} = R - n_A$. Furthermore, such a condition on β guarantees at the same time that the agents from the larger group N_A maximize their utilities when they invest with full intensity in all different-type agents as well, i.e., $y_{AA} = R - n_B$, independently of the choice y_{BB} of the agents from the smaller group. The additive nature of the welfare function $v(g(\hat{x}))$ leads then to that the derived condition is sufficient to guarantee that the value function is maximized when all agents invest with full intensity in all other same-type agents, i.e., $y_{AA} = R - n_B$ and $y_{BB} = R - n_A$.

First, fix an arbitrary quantity $y_{AA} \in [R - n_B, n_A - 1]$, and let us look for the quantity y_{BB} that solves the (hypothetical) problem

$$\max_{y_{BB} \in [R - n_A, n_B - 1]} u\Big(y_{BB}, \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_B}\Big).$$

Then, recall Assumption 2–(4) (c) establishes that $\partial u(s_i, d_i)/\partial s_i < \partial u(s_i, d_i)/\partial d_i$ for each (s_i, d_i) such that $d_i/s_i < \beta$. Therefore, if

$$\beta > \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_B y_{BB}}$$

for each $y_{BB} \in [R-n_A, n_B-1]$, then we can guarantee that the solution to the problem of this first step is uniquely given by $y_{BB} = R-n_A$. Since the function $[nR - n_A y_{AA} - n_B y_{BB}]/2n_B y_{BB}$ is strictly decreasing in y_{BB}, it follows that

$$\beta > \frac{nR - n_A y_{AA} - n_B(R-n_A)}{2n_B(R-n_A)} \tag{21}$$

is a sufficient condition that ensures that maximization of the utility of the agents $j \in N_B$ is characterized by $y_{BB} = R-n_A$, for any given $y_{AA} \in [R-n_B, n_A-1]$.

Secondly, take as given an arbitrary quantity $y_{BB} \in [R-n_A, n_B-1]$, and restrict attention to the (hypothetical) problem of finding the values of y_{AA} that solve

$$\max_{y_{AA} \in [R-n_B, n_A-1]} u\left(y_{AA}, \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_A}\right).$$

Using again Assumption 2–(4) (c), we can guarantee the solution to the problem above is characterized by $y_{AA} = R-n_B$ if

$$\beta > \frac{nR - n_A y_{AA} - n_B y_{BB}}{2n_A y_{AA}}$$

for each $y_{AA} \in [R-n_B, n_A-1]$. Since the function $[nR - n_A y_{AA} - n_B y_{BB}]/2n_A y_{AA}$ is strictly decreasing in y_{AA}, it follows that

$$\beta > \frac{nR - n_A(R-n_B) - n_B y_{BB}}{2n_A y_{BB}} \tag{22}$$

is a sufficient condition that ensures maximization of the utility of the agents $i \in N_A$ is characterized by $y_{AA} = R-n_B$, for any given $y_{BB} \in [R-n_A, n_B-1]$.

Crucially, if both conditions Eq. (21) and Eq. (22) are simultaneously satisfied for $y_{AA} = R-n_B$ and $y_{BB} = R-n_A$, then the utility of the agents from the smaller group N_B is maximized when they choose $y_{BB} = R-n_A$ conditional on the choice $y_{AA} = R-n_B$, while at the same time, the utility of the agents from the smaller group N_A is maximized when they choose $y_{AA} = R-n_B$ conditional on the choice $y_{BB} = R-n_A$. Since such sufficient conditions combined guarantee the above mentioned (common) features for the optimal choices of the two (hypothetical) problems relative to each of the populations, it follows that such conditions are sufficient to ensure that the only solution to the problem in Eq. (18) entails $y_{AA} = R-n_B$ and $y_{BB} = R-n_A$.

Note that Eq. (21) and Eq. (22) are simultaneously satisfied for $y_{AA} = R-n_B$ and $y_{BB} = R-n_A$ if and only if

$$\beta > \max\left\{\frac{n_A}{R-n_A}, \frac{n_B}{R-n_B}\right\} = \frac{n_A}{R-n_A} = \beta_h,$$

since $n_A \geq n_B$. Therefore, if $\beta > \beta_h$ then the only way in which the social planner can maximize the value function $v(g(\hat{x}))$ is by choosing

$$y_{AA} = R - n_B,\ y_{BB} = R - n_A,\ y_{AB} = z_{AB} = n_B,\ y_{BA} = z_{BA} = n_A.$$

Accordingly, for each agent $i \in N_\theta$, each type $\theta \in \Theta$, and $\theta' \neq \theta$, an efficient network $\hat{g} = g(\hat{x})$ entails $s_i(\hat{x}) = R - n_{\theta'}$ and $d_i(\hat{x}) = n_{\theta'}$. ∎

PROOF OF *Corollary 6*. Let $\hat{x}$ be a strategy profile that satisfies the necessary condition given by Proposition 4. Take $n_A = n_B = n/2$. Then, the problem that faces the social planner stated in Eq. (18) can be rewritten as

$$\begin{aligned} &\max_{\{y_{AA}, y_{BB}\}} V(y_{AA}, y_{BB}) \\ &\text{s.t.: } y_{AA} \in [R - n/2, n/2 - 1]; \\ &\qquad y_{BB} \in [R - n/2, n/2 - 1], \end{aligned} \tag{23}$$

where

$$V(y_{AA}, y_{BB}) \equiv u\big(y_{AA}, R - (1/2)(y_{AA} + y_{BB})\big) + u\big(y_{BB}, R - (1/2)(y_{AA} + y_{BB})\big).$$

Using the problem in Eq. (23), we proceed then as follows.

(i) Note that, for each type $\theta \in \Theta$, we have that $\partial V(y_{AA}, y_{BB})/\partial y_\theta > 0$ if and only if

$$\frac{\partial u\big(y_{\theta\theta}, R - (1/2)(y_{AA} + y_{BB})\big)}{\partial s_i} > \frac{\partial u\big(y_{\theta\theta}, R - (1/2)(y_{AA} + y_{BB})\big)}{\partial d_i}$$

Assumption 2–(4) (b) allows us to establish that the inequality above is satisfied if and only if

$$\beta < \frac{R - (1/2)(y_{AA} + y_{BB})}{y_{\theta\theta}}$$

or each $y_{AA}, y_{BB} \in [R - n/2, n/2 - 1]$. Then, for the symmetric choice $y_{AA} = y_{BB} = n/2 - 1$—in which each agent from each population group invests with full intensity in all other same-type fellows—to be associated to an efficient network, the required necessary and sufficient condition on the level of assortative interests takes the form

$$\beta < \frac{2R}{n-2} - 1 = \frac{2R - n + 2}{n - 2} = \beta_l.$$

(ii) For each type $\theta \in \Theta$, we have that $\partial V(y_{AA}, y_{BB})/\partial y_\theta < 0$ if and only if

$$\frac{\partial u\big(y_{\theta\theta}, R - (1/2)(y_{AA} + y_{BB})\big)}{\partial s_i} < \frac{\partial u\big(y_{\theta\theta}, R - (1/2)(y_{AA} + y_{BB})\big)}{\partial d_i}$$

It follows from Assumption 2–(4) (c) that the inequality above is satisfied if and only if

$$\beta > \frac{R-(1/2)(y_{AA}+y_{BB})}{y_{\theta\theta}}$$

for each $y_{AA}, y_{BB} \in [R-n/2, n/2-1]$. Then, for the symmetric choice $y_{AA} = y_{BB} = R-n/2$—in which each agent from each population group invests with full intensity in all different-type agents—to be associated to an efficient network, the required necessary and sufficient condition on the level of assortative interests takes the form

$$\beta > \frac{n}{2R-n} = \beta_h.$$

(iii) Consider a level of assortative interests $\beta \in (\beta_l, \beta_h)$. It follows from (i) and (ii) above that neither choices in which all agents invest with full intensity in all their same-type fellows nor choices in which they invest with full intensity in all different-type agents induce efficient networks. Now, consider symmetric aggregate investment choices $y_{AA} = y_{BB} = \hat{y}$ that give rise to partially homophilic networks that belong to the class in which all agents behave unilaterally as in [b] of Lemma 1 ([b] in Fig. 1). Such choices induce an efficient network if and only if

$$\frac{\partial u\big(y_{\theta\theta}, R-(1/2)(y_{AA}+y_{BB})\big)}{\partial s_i} = \frac{\partial u\big(y_{\theta\theta}, R-(1/2)(y_{AA}+y_{BB})\big)}{\partial d_i}$$

It follows from Assumption 2–(4) (a) that the requirement above is satisfied if and only if

$$\beta = \frac{R-\hat{y}}{\hat{y}} = \frac{n+2(R-n_A)}{2\hat{y}} - 1 \quad \Leftrightarrow \quad \hat{y} = y_{AA} = y_{BB} = \frac{R}{1+\beta} = \frac{n+2(R-n_A)}{2(1+\beta)}$$

for $\hat{y} \in (R-n/2, n/2-1)$. Finally, note that symmetric aggregate investment choices $y_{AA} = y_{BB} = \hat{y}$ are required to ensure that the condition above holds for both population groups. ∎